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B. MATHEMATICS AND MATHEMATICAL PHYSICS

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Journal of Research of the National Bureau of Standards—B. Mathematics and Mathematical Physics
Vol. 64B, No. 4, October–December 1960

Magnetic Polarizability of a Short Right Circular Conducting Cylinder [1]

T. T. Taylor [2]

(August 1, 1960)

The magnetic polarizability tensor of a short right circular conducting cylinder is calculated in the principal axes system with a uniform quasi-static but nonpenetrating applied field. One of the two distinct tensor components is derived from results already obtained in connection with the electric polarizability of short conducting cylinders. The other is calculated to an accuracy of four to five significant figures for cylinders with radius to half-length ratios of ¼, ½, 1, 2, and 4. These results, when combined with the corresponding results for the electric polarizability, are applicable to the problem of calculating scattering from cylinders and to the design of artificial dispersive media.

1. Introduction

This article considers the problem of finding the magnetic polarizability tensor β_{ij} of a short, that is, noninfinite in length, right circular conducting cylinder under the assumption of negligible field penetration. The latter condition can be realized easily with available conducting materials and fields which, although time varying, have wavelengths long compared with cylinder dimensions and can therefore be regarded as quasi-static. The approach of the present article is parallel to that employed earlier [1] [3] by the author for finding the electric polarizability of short conducting cylinders.

The magnetic polarizability tensor relates the induced magnetic dipole moment m_i to the inducing field B_j according to the equation

$$m_i = \beta_{ij} B_j, \tag{1}$$

with the cylinder situated in free space. The centered coordinate system of principal axes used in [1], in which the z-axis coincides with the axis of rotational symmetry, is also employed here. In this system, the equal transverse components β_{xx} and β_{yy} are denoted β_{tt}, and the longitudinal component β_{zz} is denoted β_{ll}. In agreement with other conventions of [1], a represents the radius of the cylinder; b, the half-length; and c, the half-diagonal or distance from center to edge. Rationalized M.K.S. units are used throughout.

The method described has been used successfully for calculating β_{tt} for cylinders with a/b ratios of ¼, ½, 1, 2, and 4. For the calculation of β_{ll}, the following relation, proved in appendix A, is invoked:

$$\mu_0 \beta_{ll} = -\frac{1}{2} \frac{a_{tt}}{\epsilon_0}, \tag{2}$$

where α_{tt} is the transverse electric polarizability treated in [1]. It is interesting that eq. (2) is valid not only for circular cylinders but for all conducting solids of revolution. Both β_{ll} and β_{tt} are negative, indicating that an artificial medium composed of short cylinder elements arranged in a lattice will be diamagnetic.

In the transverse case, where no simple relationship such as (2) is known to exist, the prob-

[1] This invited paper is based upon a portion of a thesis which has been accepted by the faculty of the California Institute of Technology in partial fulfillment of the requirements for the degree of Doctor of Philosophy.
[2] University of California, Riverside.
[3] Figures in brackets indicate the literature references at the end of this paper.

lem which presents itself is that of finding, in the external space, a static magnetic field which has no normal component at the surface of the cylinder and which reduces to a uniform field at infinity. With such a field there is associated physically a surface current density $\mathbf{j}$ on the cylinder and a field free condition in the interior. This surface current is related to the field values at the surface by:

$$\mu_0 \mathbf{j} = \mathbf{e}_n \times \mathbf{B}, \tag{3}$$

where $\mathbf{e}_n$ is the outward normal, and it gives rise to the induced magnetic dipole moment:

$$\mathbf{m} = \frac{1}{2} \int_s \mathbf{r} \times \mathbf{j} \, ds. \tag{4}$$

As with the corresponding problem involving electric polarizability, there is no known coordinate system in which a short cylinder can be treated by the conventional method of separating Laplace's equation and a different method, involving arbitrarily good approximations to the current distribution function, must be employed.

In a manner analogous to that of [1], let the cylinder be regarded merely as a geometrical construct to which the surface current distribution is firmly affixed. If the uniform (applied) component of the magnetic field is subtracted, the remaining field will be that due to the current system on the cylinder acting alone; it will be characterized by a dipole-like external field and a perfectly uniform *internal* field which is equal and opposite to the applied field. The problem may therefore be restated in terms of finding, on a finite cylindrical surface with closed plane ends, a surface current distribution which, when acting alone, generates a uniform magnetic field in the interior. The technique employed here makes use of this viewpoint and consists in setting up, on the surface, a current density function which is completely determined by a finite number of parameters, then solving for the values of these parameters so as to obtain maximal uniformity of the current field or, alternatively, maximal cancellation of the applied field, within.

The research reported here, like that related to the electric polarizability, was performed under the direction of Prof. W. R. Smythe. The author is indebted to him for many valuable discussions and to the personnel of the California Institute of Technology Computing Center for their cooperation.

2. The Transverse Problem

In this problem, let a uniform magnetic field of magnitude B be applied in the positive y-direction. The induced surface current density $\mathbf{j}$ may be regarded as equal to $\mathbf{J}\xi$ where ξ is the thickness of an infinitesmal surface layer in which the current is confined and $\mathbf{J}$ is the value of the conventional current density vector averaged over the thickness of this layer. In the quasi-static case, $\mathbf{J}$ is solenoidal and can be regarded as the curl of an equivalent averaged magnetization $\mathbf{M}$ which is also confined to the infinitesmal surface layer and which is normal to the surface at all points. On the side of the cylinder, let

$$\frac{\mu_0 M_\rho \xi}{B} = -aF(z/b) \sin \phi, \tag{5}$$

and on the upper end,

$$\frac{\mu_0 M_z \xi}{B} = -a'G(\rho/a) \sin \phi, \tag{6}$$

where F and G are dimensionless functions as yet to be determined. The configuration just described constitutes a *nonuniform magnetic shell*, that is a magnetic shell [2, p. 261] associated with a distributed current system.

Evidently the surface currents on the side are given by

$$\frac{\mu_0 j_{zs}}{B} = F(z/b)\cos\phi, \tag{7}$$

$$\frac{\mu_0 j_{\phi s}}{B} = -\frac{a}{b}F'(z/b)\sin\phi, \tag{8}$$

where the prime denotes differentiation with respect to the dimensionless argument z/b. The surface currents on the upper end are

$$\frac{\mu_0 j_{\rho e}}{B} = -\frac{a}{\rho}G(\rho/a)\cos\phi; \tag{9}$$

$$\frac{\mu_0 j_{\phi e}}{B} = G'(\rho/a)\sin\phi. \tag{10}$$

The current distribution on the lower end is equal and opposite to that on the upper end. Typical forms for the functions involved in eqs (7) through (10) are illustrated in figure 1; a sketch of the actual lines of current flow is presented in figure 2.

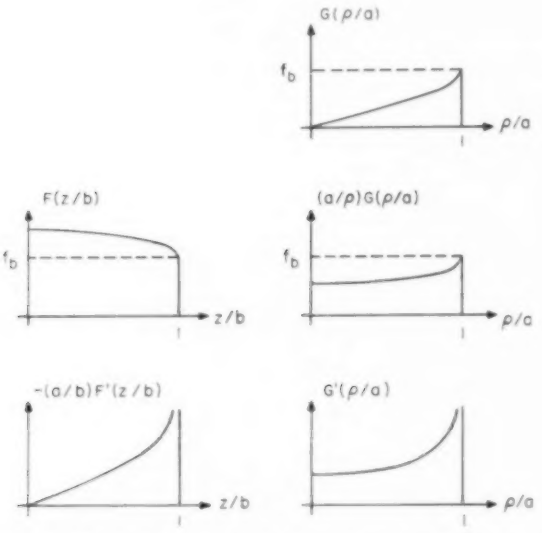

FIGURE 1. *Typical forms for the functions describing the equivalent magnetic shell and the current system.*

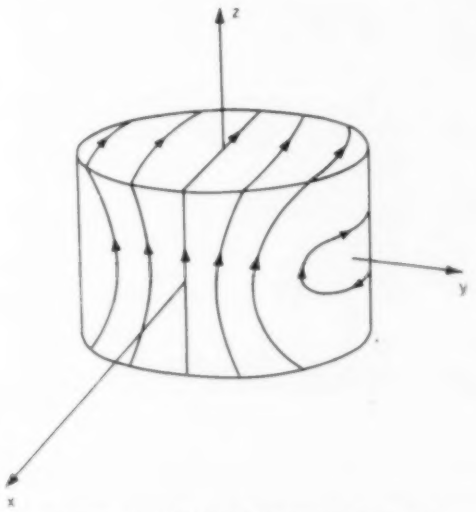

FIGURE 2. *Sketch of the current system.*

The asymptotic behavior of current and field near the edge of the cylinder is determined by the fact that, locally, the edge resembles a configuration consisting of two infinite perfectly conducting planes intersecting at right angles with the field restricted to the outer (larger) of the two angular domains. Such a configuration is a familiar one in two-dimensional potential theory. In the case at hand, two mutually perpendicular components of magnetic field are distinguished: (*a*) the component which is everywhere parallel to the edge, and (*b*) the component which bends around the edge. Each of these field components has associated with it a surface current obeying eq (3). The component of magnetic field parallel to the edge is locally uniform in the outer space and is of no further interest in the present discussion. The surface value of the component which bends around the edge, however, increases as the edge is approached and is in fact proportional to $l^{-1/3}$ where l is the perpendicular distance from

the edge to the surface point in question. This dependence, which is the same as that of the electrostatic charge density on a similar freely charged configuration, is obtained by means of the Schwarz-Christoffel transformation. The associated component of surface current is, of course, parallel to the edge and everywhere proportional to the field. On the cylinder, then, the ϕ-components of current, and hence the functions F' and G' must be asymptotic to constants times $l^{-1/3}$ as l tends to zero. This, in turn, imposes certain requirements upon F and G. Quantitatively,

$$F(z/b) \sim C_1 + C_2(b-z)^{2/3} \qquad \text{as} \quad z \to b; \tag{11}$$

$$G(\rho/a) \sim C_3 - C_4(a-\rho)^{2/3} \qquad \text{as} \quad \rho \to a; \tag{12}$$

$$-\frac{a}{b}F'(z/b) \sim \frac{2}{3}aC_2(b-z)^{-1/3} \qquad \text{as} \quad z \to b; \tag{13}$$

$$G'(\rho/a) \sim \frac{2}{3}aC_4(a-\rho)^{-1/3} \qquad \text{as} \quad \rho \to a. \tag{14}$$

It is seen that two edge conditions, $C_3 = C_1$ and $C_4 = C_2$, must be satisfied. The former of these insures continuity of current flow across the edge of the cylinder, the latter that the ϕ components on side and end match one another as both tend to infinity at the edge.

The functions F and G are represented by finite weighted sums involving the unknown coefficients f_b, f_m, g_b, and g_m, and the $\bar{\psi}_m$ functions defined in [1]. Note that the parity parameters are chosen so as to make F even and G odd and that the parameter which controls the singularity is given the value 2/3.
Thus,

$$F(z/b) = \begin{cases} f_b + \sum_{m=0}^{N_z-1} f_m \bar{\psi}_m\left(\frac{1}{2}, 0, \frac{2}{3}, \frac{z}{b}\right), & |z| \le b, \\ 0, & |z| > b, \end{cases} \tag{15}$$

and

$$G(\rho/a) = \begin{cases} g_b\frac{\rho}{a} + \sum_{m=0}^{N_z-1} g_m \bar{\psi}_m\left(1, 1, \frac{2}{3}, \frac{\rho}{a}\right) & \rho \le a, \\ 0 & \rho > a. \end{cases} \tag{16}$$

The constant terms

$$f_b = f_b \bar{\psi}_0\left(\frac{1}{2}, 0, 0, \frac{z}{b}\right), \tag{17}$$

$$g_b\frac{\rho}{a} = g_b \bar{\psi}_0\left(1, 1, 0, \frac{\rho}{a}\right), \tag{18}$$

are called the "basic" terms as in [1]. Together they constitute a very simple current system in which the flow takes place in plane rectangular loops around the cylinder, the plane of each loop being parallel to the xz-plane of figure 2. Superposed upon this basic flow pattern is another pattern described by the nonbasic (summed) terms in (15) and (16). The addition of this pattern gives a gentle curvature to the current lines and introduces the necessary singular behavior at the edges. Clearly the first edge condition demands that $g_b = f_b$ and henceforth the single symbol f_b will be used as the coefficient of the basic term in both F and G.

In relating the self field of the current system to $\mathbf{M}\xi$, a magnetic scalar potential U, defined such that $\mathbf{B} = -\nabla U$, will be used. The boundary conditions at the magnetic shell are the duals of the corresponding boundary conditions for the electric potential at a charged surface, that is, the *normal gradients* on the two sides of the shell are equal but the potential *values* differ by the local intensity of the magnetization. Quantitatively, if I and II are two adjoining

regions separated by a magnetic shell, and if $\mathbf{e}_n$ is a unit vector pointing from I into II, then

$$\frac{\partial U_{II}}{\partial n}-\frac{\partial U_{I}}{\partial n}=0; \tag{19}$$

$$U_{II}-U_{I}=\mu_0 M_n \xi. \tag{20}$$

Application of these boundary conditions and of other well-known procedures of potential theory leads to the following expression for the potential in the interior of the cylinder due to the current system on the side:

$$\frac{U_s(\rho,\phi,z)}{cB}=\frac{-a}{c}\int_0^\infty\left[\frac{2}{\pi}\int_0^b F(z'/b)\cos kz'dz'\right]kaK_1'(ka)I_1(k\rho)\cos kz\,dk\sin\phi. \tag{21}$$

The Fourier cosine transform of the F function, enclosed in the square brackets above, is easily expressed in terms of Bessel functions with the aid of eq (A9) in [1]. The function $I_1(k\rho)\cos kz\sin\phi$, which is regular in a neighborhood of the origin, may be represented in a spherical harmonic expansion about the origin. Thus (21) becomes

$$\frac{U_s(r,\theta,\phi)}{cB}=\sum_{p=0}^{\infty}\frac{-ab}{c}\left(\frac{2}{\pi}\right)^{1/2}\int_0^\infty\left[f_b\frac{J_{1/2}(kb)}{(kb)^{1/2}}+\sum_{m=0}^{N_s-1}f_m\frac{(-1)^m J_{2m+1/2+\nu'}(kb)}{(kb)^{1/2+\nu'}}\right]$$
$$kaK_1'(ka)(kc)^{2p+1}dk\,\frac{(-1)^p}{(2p+2)!}\left(\frac{r}{2}\right)^{2p+1}P^1_{2p+1}(\cos\theta)\sin\phi. \tag{22}$$

Here and elsewhere, the singularity parameter ν' for the nonbasic terms is understood to be equal to 2/3. The spherical harmonic expansion of (22) converges for all $r<a$.

With reference to the ends of the cylinder, the interior potential generated by the current distributions on the two ends is given by

$$\frac{U_e(\rho,\phi,z)}{cB}=\frac{a}{c}\int_0^\infty\left[\frac{1}{a}\int_0^a G(\rho'/a)J_1(k\rho')\rho'd\rho'\right]kae^{-kb}J_1(k\rho)\cosh kzdk\sin\phi. \tag{23}$$

This, in turn, may also be expressed as an expansion in spherical harmonics which converges for all $r<b$:

$$\frac{U_e(r,\theta,\phi)}{cB}=\sum_{p=0}^{\infty}\frac{a^2}{c}\int_0^\infty\left[f_b\frac{J_2(ka)}{(ka)}+\sum_{m=0}^{N_e-1}g_m\frac{(-1)^m J_{2m+2+\nu'}(ka)}{(ka)^{1+\nu'}}\right]$$
$$kae^{-kb}(kc)^{2p+1}dk\,\frac{1}{(2p+2)!}\left(\frac{r}{c}\right)^{2p+1}P^1_{2p+1}(\cos\theta)\sin\phi. \tag{24}$$

Evidently the sum of (22) and (24) constitutes an expansion of the interior potential due to the total current distribution, convergent within the largest sphere which can be inscribed in the cylinder. Such an expansion may also be expressed in the form

$$\frac{U(r,\theta,\phi)}{cB}=\sum_{p=0}^{\infty}\left[f_bZ_p^{sb}+\sum_{m=0}^{N_s-1}f_mZ_p^{sm}+f_bZ_p^{eb}+\sum_{m=0}^{N_e-1}g_mZ_p^{em}\right]\left(\frac{r}{c}\right)^{2p+1}P^1_{2p+1}(\cos\theta)\sin\phi, \tag{25}$$

where the Z's are obtained from (22) and (24) by integrating [3, p. 137] over k. These Z's become the matrix elements in a system of simultaneous equations which may be solved for the f_b, f_m, and g_m such that the coefficient of $(r/c)^{2p+1}P^1_{2p+1}(\cos\theta)\sin\phi$ will be unity for $p=0$ and zero for as many $p>0$ as possible. The second edge condition, also included in the system of simultaneous equations, is simply

$$\sum_{m=0}^{N_s-1}f_m+\left(\frac{a}{b}\right)^{-2/3}\sum_{m=0}^{N_e-1}g_m=0. \tag{26}$$

If one makes use of the fact that $-kaK_1'(ka)=kaK_0(ka)+K_1(ka)$, the expression for the matrix element Z_P^{sm} becomes

$$Z_p^{sm}=\frac{ab}{c}\left(\frac{2}{\pi}\right)^{1/2}\frac{(-1)^{m+p}}{(2p+2)!}\int_0^\infty [kaK_0(ka)+K_1(ka)]\,\frac{J_{2m+\frac{1}{2}+\nu'}(kb)}{(kb)^{1/2+\nu'}}\,(kc)^{2p+1}dk. \tag{27}$$

Integration yields the following result:

$$Z_p^{sm}=\frac{(-1)^{m+p}2^{2p+1-\nu'}\Gamma(m+p+\frac{3}{2})\Gamma(m+p+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(2p+3)\Gamma(2m+\frac{3}{2}+\nu')}\,\frac{a^2}{bc}\left(\frac{b}{c}\right)^{2m+2}\left[\left(m+p+\frac{1}{2}\right)\right.$$
$$\left.{}_2F_1\left(m+p+\frac{3}{2},\ m-p+\nu';\ 2m+\frac{3}{2}+\nu';\frac{b^2}{c^2}\right)+\frac{1}{2}\,{}_2F_1\left(m+p+\frac{3}{2},\ m-p+1+\nu';\ 2m+\frac{3}{2}+\nu';\frac{b^2}{c^2}\right)\right]. \tag{28}$$

The matrix element Z_P^m is simpler:

$$Z_p^{sm}=\frac{a^2}{c}\,\frac{(-1)^m}{(2p+2)!}\int_0^\infty e^{-kb}\,\frac{J_{2m+2+\nu'}(ka)}{(ka)^{1+\nu'}}\,(kc)^{2p+1}kadk. \tag{29}$$

This becomes

$$Z_p^{sm}=\frac{(-1)^m2^{2p+1-\nu'}\Gamma(m+p+\frac{5}{2})\Gamma(m+p+2)}{\Gamma(\frac{1}{2})\Gamma(2p+3)\Gamma(2m+3+\nu')}\,\frac{b}{c}\left(\frac{a}{c}\right)^{2m+4}$$
$${}_2F_1\left(m+p+\frac{5}{2},\ m-p+1+\nu';\ 2m+3+\nu';\frac{a^2}{c^2}\right). \tag{30}$$

The basic matrix elements, Z_p^{sb} and Z_p^{eb}, are obtained by setting both m and ν' equal to zero in the preceding formulas.
Thus,

$$Z_p^{sb}=\frac{(-1)^p2^{2p+1}\Gamma(p+\frac{3}{2})\Gamma(p+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(2p+3)\Gamma(\frac{3}{2})}\,\frac{a^2b}{c^3}\left[\left(p+\frac{1}{2}\right)\right.$$
$$\left.{}_2F_1\left(p+\frac{3}{2},-p;\frac{3}{2};\frac{b^2}{c^2}\right)+\frac{1}{2}\,{}_2F_1\left(p+\frac{3}{2},-p+1;\frac{3}{2};\frac{b^2}{c^2}\right)\right]; \tag{31}$$

$$Z_p^{eb}=\frac{2^{2p+1}\Gamma(p+\frac{5}{2})\Gamma(p+2)}{\Gamma(\frac{1}{2})\Gamma(2p+3)\Gamma(3)}\,\frac{a^4b}{c^5}\,{}_2F_1\left(p+\frac{5}{2},-p+1;\ 3;\frac{a^2}{c^2}\right). \tag{32}$$

Using the appropriate formulas for analytic continuation [4, p. 108], one obtains

$${}_2F_1\left(p+\frac{5}{2},-p+1;\ 3;\frac{a^2}{c^2}\right)=\frac{\Gamma(3)\Gamma(-\frac{1}{2})}{\Gamma(-p+\frac{1}{2})\Gamma(p+2)}\,{}_2F_1\left(p+\frac{5}{2},-p+1;\frac{3}{2};\frac{b^2}{c^2}\right)$$
$$+\frac{c}{b}\,\frac{\Gamma(3)\Gamma(\frac{1}{2})}{\Gamma(p+\frac{5}{2})\Gamma(-p+1)}\,{}_2F_1\left(-p+\frac{1}{2},p+2;\frac{1}{2};\frac{b^2}{c^2}\right). \tag{33}$$

By [4, p. 3],

$$\frac{\Gamma(-\frac{1}{2})}{\Gamma(-p+\frac{1}{2})}=\frac{-(-1)^p\Gamma(p+\frac{1}{2})}{\Gamma(\frac{3}{2})}. \tag{34}$$

The second term in (33) vanishes unless $p=0$, in which case the hypergeometric function is easily summed:

$${}_2F_1\left(\frac{1}{2},\ 2;\ \frac{1}{2};\frac{b^2}{c^2}\right)=\left(1-\frac{b^2}{c^2}\right)^{-2}=\left(\frac{c}{a}\right)^4. \tag{35}$$

Let the symbol δ_{0p} be equal to unity for $p=0$ and to zero for $p\neq0$; then

$$Z_p^{eb}=\delta_{0p}-\frac{(-1)^p2^{2p+1}\Gamma(p+\frac{5}{2})\Gamma(p+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(2p+3)\Gamma(\frac{3}{2})}\,\frac{a^4b}{c^5}\,{}_2F_1\left(p+\frac{5}{2},-p+1;\frac{3}{2};\frac{b^2}{c^2}\right). \tag{36}$$

In (25), the total coefficient of f_b is the sum of Z_p^{ab} and Z_p^{eb}. This sum is denoted by Z_p^b and is given by

$$Z_p^b=\delta_{0p}+\frac{(-1)^p 2^{2p+1}\Gamma(p+\frac{3}{2})\Gamma(p+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(2p+3)\Gamma(\frac{3}{2})}\frac{a^2b}{c^3}\left[\left(p+\frac{1}{2}\right){}_2F_1\left(p+\frac{3}{2},-p;\frac{3}{2};\frac{b^2}{c^2}\right)\right.$$
$$\left.+\frac{1}{2^2}F_1\left(p+\frac{3}{2},-p+1;\frac{3}{2};\frac{b^2}{c^2}\right)-\frac{a^2}{c^2}\left(p+\frac{3}{2}\right){}_2F_1\left(p+\frac{5}{2},-p+1;\frac{3}{2};\frac{b^2}{c^2}\right)\right]. \quad (37)$$

Using the third Gauss relationship, as listed in [4, p. 103]:

$$Z_p^b=\delta_{0p}-\frac{(-1)^p 2^{2p}\Gamma(p+\frac{3}{2})\Gamma(p+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(2p+3)\Gamma(\frac{3}{2})}\frac{a^2b}{c^3}\,{}_2F_1\left(p+\frac{3}{2},-p+1;\frac{3}{2};\frac{b^2}{c^2}\right). \quad (38)$$

This has an even simpler form, obtained by an elementary transformation of the hypergeometric function and a simplification of the gamma function multiplier:

$$Z_p^b=\delta_{0p}-\frac{(-1)^p\Gamma(p+\frac{1}{2})}{2\Gamma(\frac{1}{2})\,\Gamma(p+2)}\frac{b}{c}\,{}_2F_1\left(p+\tfrac{1}{2},-p;\tfrac{3}{2};\frac{b^2}{c^2}\right). \quad (39)$$

The three formulas, (28), (30), and (39), contain the information necessary for a numerical calculation of the matrix elements. If both the electric and the magnetic polarizabilities are being calculated for the same cylinder, much labor can be saved by comparing the expressions for the matrix elements in this paper with those in [1] and exploiting the similarities.

The magnetic dipole moment, given by eq (4), has only a y-component,

$$m_y=\frac{1}{2}\int_{-b}^{b}\int_0^{2\pi}(-zj_{\phi s}\sin\phi-aj_{zs}\cos\phi)ad\phi dz+\int_0^a\int_0^{2\pi}b(j_{\rho e}\cos\phi-j_{\phi e}\sin\phi)d\phi\rho d\rho. \quad (40)$$

Using expressions (7) through (10),

$$\frac{\mu_0 m_y}{B}=\frac{1}{2}\int_{-b}^{b}\int_0^{2\pi}\left(z\frac{a}{b}F'\sin^2\phi-aF\cos^2\phi\right)ad\phi dz+\int_0^a\int_0^{2\pi}b\left(-\frac{a}{\rho}G\cos^2\phi-G'\sin^2\phi\right)d\phi\rho d\rho; \quad (41)$$

$$\frac{\mu_0 m_y}{B}=\pi a^2 l\left[\int_0^1(uF'-F)du-\int_0^1(G+uG')du\right]. \quad (42)$$

By parts,

$$\frac{\mu_0 m_y}{B}=\pi a^2 b\left\{[uF]_0^1-2\int_0^1 Fdu-[uG]_0^1\right\}. \quad (43)$$

The first edge condition requires that $F(1)=G(1)$; hence,

$$\frac{\mu_0 m_y}{B}=-2\pi a^2 b\int_0^1 Fdu. \quad (44)$$

The quantity m_y/B is the transverse polarizability and the geometrical volume v_0 is easily recognized. The orthogonality relationships given in [1] aid in the integration of F, yielding the following formula for the polarizability:

$$\frac{\mu_0\beta_{tt}}{v_0}=-\left[f_b+\frac{\Gamma(\frac{1}{2})}{2^{5/3}\Gamma(\frac{1}{6})}f_0\right]. \quad (45)$$

3. Results

The two distinct tensor components of the magnetic polarizability are given in table 1. Actual values of the expansion coefficients for the F and G functions, which describe the surface current distribution via the equivalent magnetic shell for the transverse case, are included in appendix B. These data, upon which the β_{tt} results of table 1 are based, were obtained by solving sets of 18 simultaneous equations. Experimentation with fewer equations has led the author to the conclusion that the accuracy is not quite as good as that obtained for the electric polarizability in [1]. In other words, the β_{tt} figures have a probable error of about 2 units in the least significant digit for a/b equal to unity and about 10 units in this digit for a/b equal to 4 or to $\frac{1}{4}$. The β_{ll} figures, on the other hand, were obtained from the electric polarizabilities simply by the application of eq (2), and therefore enjoy the same accuracy belonging to the latter.

TABLE 1

a/b	$\mu_0\beta_{ll}/v_0$	$\mu_0\beta_{tt}/v_0$
0	−1.0000	−2.0000
$\frac{1}{4}$	−1.1575	−1.8506
$\frac{1}{2}$	−1.3057	−1.7351
1	−1.5853	−1.5795
2	−2.1087	−1.4140
4	−3.0907	−1.2716
∞	$-\infty$	−1.0000

$v_0=2\pi a^2 b$

As an additional check upon the numerical calculations in the transverse problem, a procedure analogous to that employed in [1] was used. Consider the total field $\mathbf{B}_t$ which is the sum of the applied field and the field due to the current system on the cylinder. A normal component B_{tn} of the total field at any point on the surface of the cylinder indicates that an error is present. If n represents distance away from the surface in the outward normal direction at a particular surface point, one may calculate the displacement of the surface Δn which, considering the local value of the normal field gradient, would reduce the normal field component to zero. In other words, let the following equation be solved for Δn:

$$B_{tn}+\frac{\partial B_{tn}}{\partial n}\,\Delta n=0. \tag{46}$$

This Δn may then be compared with the gross dimensions of the cylinder. The local field gradient $\partial B_{tn}/\partial n$ depends upon the current distribution at the point in question and may be calculated by appealing to eq (3) and to the solenoidal character of $\mathbf{B}_t$. To first order, one has for any point on the side of the cylinder

$$\frac{\partial B_{t\rho}}{\partial \rho}=\mu_0\left(-\frac{1}{a}\,\frac{\partial j_{zs}}{\partial \phi}+\frac{\partial j_{\phi s}}{\partial z}\right); \tag{47}$$

$$\frac{\partial B_{t\rho}}{\partial \rho}=\frac{B\sin\phi}{a}\left(F-\frac{a^2}{b^2}\,F''\right). \tag{48}$$

Let the ρ-component of the field due to the current system alone be denoted B_ρ. Then $B_{t\rho}=B_\rho+B\sin\phi$, $\Delta n=\Delta\rho$, and one obtains for the latter,

$$\frac{\Delta\rho}{a}=\frac{\dfrac{B_\rho}{B\sin\phi}+1}{-F+\dfrac{a^2}{b^2}\,F''}. \tag{49}$$

The quantities B_ρ, F, F'' and finally $\Delta\rho/a$ were calculated for $z=0$, that is for points on the "equator" of the cylinder; hence the term "equator check" in appendix B.

The same method applied to the upper end of the cylinder yields

$$\frac{\partial B_{tz}}{\partial z}=\mu_0\left(-\frac{j_{\phi e}}{\rho}-\frac{\partial j_{\phi e}}{\partial \rho}+\frac{1}{\rho}\frac{\partial j_{\rho e}}{\partial \phi}\right), \tag{50}$$

$$\frac{\partial B_{tz}}{\partial z}=\frac{B\sin\phi}{a}\left(-\frac{G'}{u}-G''+\frac{G}{u^2}\right), \tag{51}$$

where u represents ρ/a. Substitution of the expansion $A_1u+A_3u^3+A_5u^5+\ldots$ for G, which is known to be an odd function, gives

$$\frac{\partial B_{tz}}{\partial z}=\frac{B\sin\phi}{a}\left(-8A_3u-24A_5u^3+\ldots\right). \tag{52}$$

Since both sides of (52) vanish at $\rho=0$, that is at the "pole" of the cylinder, one must differentiate with respect to ρ in order to obtain something useful. Thus,

$$\frac{\partial^2 B_{tz}}{\partial\rho\partial z}=\frac{B\sin\phi}{a^2}\left(-8A_3\right)=\frac{-4B\sin\phi}{3a^2}G'''(0). \tag{53}$$

Furthermore, for any point on the upper end, B_{tz} is simply B_z, the field due to the current system alone. Finally,

$$\frac{\Delta z}{b}=\frac{3a^2}{4bG'''(0)}\frac{\partial}{\partial\rho}\left(\frac{B_z}{B\sin\phi}\right). \tag{54}$$

Values of $\partial B_z/\partial\rho$, G''', and $\Delta z/b$ were calculated for the pole; data are given under the heading of "pole check" in appendix B. Generally speaking, the pole checks are satisfactory but are not as good as the corresponding checks in [1].

4. Appendix A: Relation Between the Longitudinal Magnetic and the Transverse Electric Polarizabilities of a Conducting Solid of Revolution

Let figure 3 illustrate a typical solid of revolution formed by rotating the curve $\rho=\rho(z)$ about the z-axis.

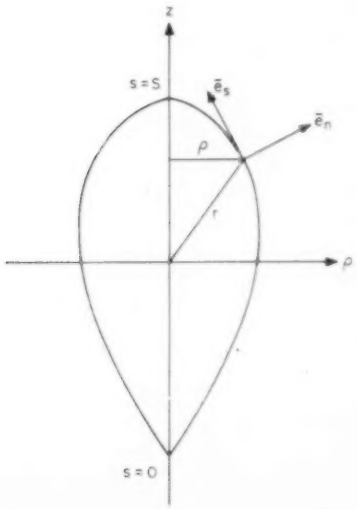

FIGURE 3. *Cross section of a typical solid of revolution showing coordinates and unit vectors.*

Let the unit vectors $\mathbf{e}_s$ and $\mathbf{e}_n$ point along the tangent and the outward normal of this curve, respectively. The variable s, associated with $\mathbf{e}_s$, measures distance along the curve and encompasses the solid as it varies over the interval $0 \leq s \leq S$. It is sufficient if $\rho(z)$ has a uniquely defined tangent and is normal at almost every point in this interval.

If the applied fields consist of a uniform magnetic field B in the z-direction and a uniform electric field E in the x-direction, the following is to be proved:

$$\frac{\mu_0 m_z}{B} = -\frac{1}{2}\frac{p_x}{\epsilon_0 E}. \tag{A1}$$

In the electric case, the scalar potential satisfies the conditions:

$$\left.\begin{aligned} &V = -Ef_1(\rho, z)\cos\phi; \\ &f_1 = 0 \quad \text{on the surface of the solid}; \\ &f_1 \sim \rho \quad \text{at infinity}; \\ &\nabla^2 V = 0. \end{aligned}\right\} \tag{A2}$$

The magnetic case, on the other hand, is analyzed with the aid of the vector potential which, by symmetry, has only a ϕ-component. This component must vanish at $s=0$, $s=S$, and at all intermediate surface points; otherwise there would be a normal component of magnetic field at the surface. One finds that

$$\left.\begin{aligned} &\mathbf{A} = \mathbf{e}_\phi \frac{B}{2} f_2(\rho, z); \\ &f_2 = 0 \quad \text{on the surface of the solid}; \\ &f_2 \sim \rho \quad \text{at infinity}; \\ &\nabla^2 \mathbf{A} = 0. \end{aligned}\right\} \tag{A3}$$

Application of the Laplacian operator to either $\cos\phi\, f_1$ or to $\mathbf{e}_\phi f_2$ yields the same equation in ρ and z. In other words,

$$\left\{\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} - \frac{1}{\rho^2} + \frac{\partial^2}{\partial z^2}\right\}\left\{\begin{matrix} f_1 \\ f_2 \end{matrix}\right\} = 0. \tag{A4}$$

Since f_1 and f_2 obey the same differential equation, have the same boundary conditions at the surface of the solid, and have identical asymptotic behavior at infinity, it is clear that $f_1 = f_2 = f$. The surface charge density in the electric case is

$$\sigma = -\epsilon_0 \mathbf{e}_n \cdot \nabla V = \epsilon_0 E \cos\phi\ (\mathbf{e}_n \cdot \nabla f), \tag{A5}$$

and it is easily shown that the x-component of the induced electric dipole moment is

$$p_x = \epsilon_0 E \pi \int_0^S (\mathbf{e}_n \cdot \nabla f)\, \rho^2 ds. \tag{A6}$$

The surface current in the magnetic case as given by (3) is:

$$\mathbf{j} = \frac{1}{\mu_0}\mathbf{e}_n \times (\nabla \times \mathbf{A}) = \frac{B}{2\mu_0}\mathbf{e}_n \times (\nabla \times \mathbf{e}_\phi f). \tag{A7}$$

On the surface, where $f=0$, this reduces to the following:

$$\mathbf{j}=-\frac{B}{2\mu_0}\mathbf{e}_\phi(\mathbf{e}_n\cdot\nabla f), \tag{A8}$$

and the magnetic dipole moment is obviously

$$m_z=-\frac{B\pi}{\mu_0 2}\int_0^S(\mathbf{e}^n\cdot\nabla f)\rho^2 ds. \tag{A9}$$

A comparison of (A9) with (A6) shows that the asserted relation (A1) is true.

5. Appendix B: Values of the Expansion Coefficients for the Functions F and G

Values of the expansion coefficients for the functions F and G and the results of the checking procedure at pole and equator are given in tables 2 through 6. The notation "$A(p)$" means "A times 10^p."

TABLE 2

$a/b=\frac{1}{4}$

m	f_m	g_m
basic	+0.13640434 (+1)	
0	+0.94335081 (+0)	−0.37183474 (+0)
1	+0.47094227 (+0)	−0.18563192 (+0)
2	+0.15962976 (+0)	−0.97767116 (−1)
3	+0.52222212 (−1)	
4	+0.17272056 (−1)	
5	+0.55233089 (−2)	
6	+0.16203994 (−2)	
7	+0.41694492 (−3)	
8	+0.90138919 (−4)	
9	+0.15575188 (−4)	
10	+0.20048427 (−5)	
11	+0.17022684 (−6)	
12	+0.71319103 (−8)	
Equator check	$B_\rho/B\sin\phi=-0.99999999$ $\Delta\rho/a$: negligible	
Pole check	$a\frac{\partial}{\partial\rho}(B_z/B\sin\phi)=+0.11001902$ $\Delta z/b=+0.01246495$	

TABLE 3

$a/b=\frac{1}{2}$

m	f_m	g_m
basic	+0.13390263 (+1)	
0	+0.76790238 (+0)	−0.32481666 (+0)
1	+0.23079152 (+0)	−0.14321008 (+0)
2	+0.58147254 (−1)	−0.97128394 (−1)
3	+0.16764875 (−1)	−0.67455522 (−1)
4	+0.50092196 (−2)	−0.36913235 (−1)
5	+0.13805064 (−2)	−0.11074155 (−1)
6	+0.32120205 (−3)	
7	+0.57675668 (−4)	
8	+0.70016001 (−5)	
9	+0.42707647 (−6)	
Equator check	$B_\rho/B\sin\phi=-1.00000004$ $\Delta\rho/a$: negligible	
Pole check	$a\frac{\partial}{\partial\rho}(B_z/B\sin\phi)=-0.00614087$ $\Delta z/b=-0.02256388$	

TABLE 4

$a/b=1$

m	f_m	g_m
basic	+0.12992927 (+1)	
0	+0.54323936 (+0)	−0.29402467 (+0)
1	+0.92098198 (−1)	−0.12410492 (+0)
2	+0.17619634 (−1)	−0.80893285 (−1)
3	+0.38217146 (−2)	−0.59953649 (−1)
4	+0.67793221 (−3)	−0.44063915 (−1)
5	+0.66769575 (−4)	−0.29238560 (−1)
6		−0.16182197 (−1)
7		−0.68464716 (−2)
8		−0.19415309 (−2)
9		−0.27441144 (−3)
Equator check	$B_\rho/B\sin\phi=-0.99999998$ $\Delta\rho/a$: negligible	
Pole check	$a\frac{\partial}{\partial\rho}(B_z/B\sin\phi)=-0.00000485$ $\Delta z/b=-0.00001512$	

TABLE 5

$a/b=2$

m	f_m	g_m
basic	+0.12407197 (+1)	
0	+0.33590275 (+0)	−0.25652350 (+0)
1	+0.30150397 (−1)	−0.11210208 (+0)
2	+0.28485520 (−2)	−0.72413481 (−1)
3		−0.52386398 (−1)
4		−0.37934754 (−1)
5		−0.25696044 (−1)
6		−0.15530579 (−1)
7		−0.80562157 (−2)
8		−0.34498633 (−2)
9		−0.11632626 (−2)
10		−0.28842547 (−3)
11		−0.46632080 (−4)
12		−0.36819245 (−5)
Equator check	$B_\rho/B\sin\phi=-1.00000288$ $\Delta\rho/a=+0.00000115$	
Pole check	$a\frac{\partial}{\partial\rho}(B_z/B\sin\phi)=-0.00000013$ $\Delta z/b=-0.00000100$	

TABLE 6

$a/b=4$

m	f_m	g_m
basic	+0.11775253 (+1)	
0	+0.18232541 (+0)	−0.21277158 (+0)
1	+0.76983950 (−3)	−0.98721317 (−1)
2		−0.61346284 (−1)
3		−0.39649681 (−1)
4		−0.24392554 (−1)
5		−0.13622852 (−1)
6		−0.66941775 (−2)
7		−0.28211792 (−2)
8		−0.99369454 (−3)
9		−0.28363133 (−3)
10		−0.62898095 (−4)
11		−0.10154428 (−4)
12		−0.10698227 (−5)
13		−0.53798542 (−7)
Equator check	$B\rho/B\sin\phi=-1.00378633$	
	$\Delta\rho/a=+0.00094822$	
Pole check	$a\frac{\partial}{\partial\rho}(B_z/B\sin\phi)=+0.00000000$	
	$\Delta z/b$: negligible	

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Accuracy of Monte Carlo Methods in Computing Finite Markov Chains

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Experiments are made with the Markov chain presented by the children's game of Chutes and Ladders. Statistics, such as the average length of play, are computed on the IBM 704 from 2^{15} simulated plays of the game. These Monte Carlo results are then compared with the "exact solution" obtained by powering the matrix of transition probabilities. Convergence is shown to obey the familiar "$N^{-1/2}$" law.

1. Introduction

Since Monte Carlo methods are generally applied to problems that are too difficult for other types of analysis there is seldom opportunity to compare their results with "exact solutions." In this paper experiments are described with the Markov chain presented by the children's game of Chutes and Ladders.[1] In particular, the calculation of the average length $\overline{L}$ of a play of this game is considered. An exact value for $\overline{L}$ is found according to Markov chain theory. The Monte Carlo calculation for $\overline{L}$ involves 2^{15} simulated plays of the game. Furthermore, the probabilities of being in a given state by a fixed number of throws are computed theoretically and experimentally.

It is shown that well-known theorems of statistics imply that the Monte Carlo results should converge to the true solutions by "$N^{-1/2}$" laws. Numerical values are shown to follow these laws. All work was carried out on the IBM 704 computer.

2. Description of the Game

First the game of Chutes and Ladders is described in detail. The game is considered to be played in its solitaire form. The game involves a playing piece, one die, and a playing board which has squares numbered from 1 to 100. Certain pairs of these 100 squares are connected by "chutes" and others are connected by "ladders." The purpose of the chutes is to delay the game by forcing the player to return to the square at the bottom of a chute if he should land on the square at its top. On the other hand, the ladders speed up the game by permitting a player to advance directly from the square at the bottom of a ladder to the square at its top, and thus bypass the squares in between. There are 10 chutes connecting the pairs of numbered squares (16,6), (47,26), (49,11), (56,53), (62,19), (64,60), (87,24), (93,73), (95,75), and (98,78). However, nine ladders connect the squares (1,38), (4,14), (9,31), (21,42), (28,84), (36,44), (51,67), (71,91), and (80,100).

[1] Copyright Milton-Bradley Co.

Since the playing piece can never remain at either the top of a chute or the bottom of a ladder, there are only 81 squares at which it can stop.

The playing piece is off the board at the start of the game and the player tosses the die so as to determine a number from 1 to 6. He then moves his playing piece through the same number of squares as the number on the die. Should his playing piece stop at the bottom of a ladder he must immediately move to the top of the ladder. He then repeats the throw of the die and moves accordingly. If at any time he lands at the top of a chute he must proceed directly to its bottom before the next die toss. The game is concluded when the playing piece reaches square 100. The board actually has two winning positions, since there is a ladder from square 80 to square 100. One requirement is that square 100 be reached by an exact throw of the die.

3. Theoretical Solution

The game can be interpreted as a finite Markov chain with 82 states. These states are indexed by $i=0, 1, \ldots, 81$, where the state $i=0$ corresponds to the starting position and the state $i=81$ corresponds to the playing piece being at square 100. All squares except those at the top of a chute or the bottom of a ladder have a corresponding state in the chain.

Let $X_i(n)$, $i=0, 1, \ldots, 81$ represent the probability of being in state i after n throws of the die. Then $X_0(0)=1$ and $X_0(n)=0$ for $n>0$. Also, $\sum_{i=0}^{81} X_i(n)=1$ for every $n\geq 0$. Let p_{ij} be the conditional probability of moving from square i to square j on one die toss whenever square i is occupied. We see immediately that

$$\begin{bmatrix}\text{Prob of state}\\ j \text{ on } n \text{ tosses}\end{bmatrix} = \sum_{i=0}^{81} \begin{bmatrix}\text{Prob of transition}\\ \text{from state } i \text{ to}\\ \text{state } j \text{ on one toss}\end{bmatrix} \cdot \begin{bmatrix}\text{Prob of state } i\\ \text{on } n-1 \text{ tosses}\end{bmatrix} \quad (3.1)$$

In other words, the elements $X_j(n)$ satisfy the relations

$$X_j(n)=\sum_{i=0}^{81} p_{ij}X_i(n-1), \qquad \begin{matrix}0\le j\le 81\\ n=1,2,3,\ \ldots\end{matrix} \tag{3.2}$$

where $X_i(0)=\begin{cases}0, & 1\le i\le 81\\ 1, & i=0\end{cases}$. If we let $X(n)$ be a row vector whose components are $X_j(n)$ we can write (3.2) in the form

$$X(n)=X(n-1)P=X(0)P^n \tag{3.3}$$

where $X(0)$ is the vector whose components are $X_0(0), X_1(0), \ldots, X_{81}(0)$ and P^n is the nth power of

$$P=\begin{bmatrix} p_{00}, & p_{01}, & \ldots, & p_{0,81}\\ \cdot & & & \cdot\\ \cdot & & & \cdot\\ \cdot & & & \cdot\\ p_{81,0}, & p_{81,1}, & \ldots, & p_{81,81}\end{bmatrix}.$$

The matrix elements p_{ij} will be called the transition probabilities; certain of them are given in table 1.

We now consider the average length, $\overline{L}$, of the game. This may be computed from the expression

$$\overline{L}=\sum_{n=1}^{\infty} n[X_{81}(n)-X_{81}(n-1)], \tag{3.4}$$

where the term $X_{81}(n)-X_{81}(n-1)$ represents the probability of reaching the final square on die toss number n.

An alternative expression which can be used for computing $\overline{L}$ is given as follows. First introduce the truncated matrix P_T formed by crossing out both the last row and the last column of P. Then form the matrix

$$T=(I-P_T)^{-1} \tag{3.5}$$

TABLE 1. *Partial listing of matrix elements p_{ij}*

i \ j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	. . .	69	70	71	72	73	74	75	76	77	78	79	80	81
0		1/6	1/6	1/6	1/6							1/6																	
1			1/6	1/6	1/6	1/6	1/6					1/6																	
2				1/6	1/6	1/6	1/6					1/6																	
3					1/6	1/6	1/6	1/6	1/6																				
4						1/6	1/6	1/6	1/6	1/6																			
5							1/6	1/6	1/6	1/6	1/6																		
6								1/6	1/6	1/6	1/6	1/6																	
7				1/6					1/6	1/6	1/6	1/6	1/6																
8				1/6						1/6	1/6	1/6	1/6	1/6															
9				1/6							1/6	1/6	1/6	1/6	1/6														
10				1/6								1/6	1/6	1/6	1/6														
11				1/6									1/6	1/6	1/6														
12				1/6										1/6	1/6														
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.																													
.																													
66																	1/6	1/6	1/6										
67																	1/6	1/6	1/6	1/6									
68																	1/6	1/6	1/6	1/6	1/6								
69																		1/6	1/6	1/6	1/6	1/6							
70																			1/6	1/6	1/6	1/6	1/6						
71																				1/6	1/6	1/6	1/6	1/6					
72																					1/6	1/6	1/6	1/6	1/6				
73																						1/6	1/6	1/6	1/6				
74																							1/6	1/6	1/6	1/6			
75																								1/6	1/6	1/6	1/6		
76																									1/6	1/6	1/6		
77																										1/6	1/6	1/6	1/6
78																										1/3	1/6	1/6	1/6
79																											1/2	1/6	1/6
80																												5/6	1/6
81																													1

with elements T_{ij}. It is shown on p. 329 of Kemeny, Snell, and Thompson [2] that $\bar{L}$ is given by

$$\bar{L}=\sum_{j=1}^{80} T_{0j}, \tag{3.6}$$

which is simply the sum of the first row of the matrix $(I-P_T)^{-1}$.

Equation (3.3) shows that a direct numerical solution by successive matrix vector multiplications can be used to calculate the vectors $X(n)$. This solution may not be feasible for full matrices or for chains with a large number of states, since powering the matrix may be subject to space and time limitations and roundoff error. However, for Chutes and Ladders only 476 of the 6,724 transition probabilities do not vanish. The $X(n)$ were easily calculated on the IBM computer by taking the sums in (3.2) over the nonvanishing elements. The time required for each matrix vector multiplication was about 2 sec. To illustrate the results we have presented the vectors $X(7)$ and $X(31)$ in table 2. The vector $X(7)$ is of interest since the computations showed that $X_{81}(n)=0$ for $n<7$, so that the game cannot be completed in fewer than seven throws; the vector $X(31)$ is of interest since $X_{81}(31)$ has also been computed by Monte Carlo.

TABLE 2. *The vectors $X(7)$ and $X(31)$*

i	$X_i(7)$	$X_i(31)$	i	$X_i(7)$	$X_i(31)$
0	0	0	41	.15936×10⁻¹	.62209×10⁻²
1	0	0	42	.88735×10⁻²	.37065×10⁻²
2	0	0	43	.16175×10⁻¹	.74137×10⁻²
3	0	0	44	.95950×10⁻²	.42890×10⁻²
4	.19740×10⁻¹	.32552×10⁻²	45	.79733×10⁻²	.37599×10⁻²
5	.47725×10⁻²	.56622×10⁻³	46	.71266×10⁻²	.33331×10⁻²
6	.60942×10⁻²	.66476×10⁻³	47	.81697×10⁻²	.39130×10⁻²
7	.76982×10⁻²	.78046×10⁻³	48	.78589×10⁻²	.39497×10⁻²
8	.28607×10⁻¹	.81893×10⁻²	49	.92057×10⁻²	.73498×10⁻²
9	.14518×10⁻¹	.23394×10⁻²	50	.50654×10⁻²	.38812×10⁻²
10	.13064×10⁻¹	.21801×10⁻²	51	.40509×10⁻²	.39027×10⁻²
11	.14489×10⁻¹	.24607×10⁻²	52	.20898×10⁻²	.33210×10⁻²
12	.16036×10⁻¹	.27729×10⁻²	53	.12610×10⁻²	.32115×10⁻²
13	.17811×10⁻¹	.31194×10⁻²	54	.12867×10⁻¹	.74863×10⁻²
14	.17300×10⁻¹	.22388×10⁻²	55	.21219×10⁻²	.31170×10⁻²
15	.22630×10⁻¹	.61242×10⁻²	56	.24970×10⁻²	.36590×10⁻²
16	.21066×10⁻¹	.29082×10⁻²	57	.27328×10⁻²	.36162×10⁻²
17	.19726×10⁻¹	.25041×10⁻²	58	.31579×10⁻²	.39672×10⁻²
18	.24366×10⁻¹	.28399×10⁻²	59	.71766×10⁻²	.11658×10⁻¹
19	.30696×10⁻¹	.12958×10⁻¹	60	.26578×10⁻²	.44668×10⁻²
20	.26285×10⁻¹	.47659×10⁻²	61	.55870×10⁻²	.12502×10⁻¹
21	.50694×10⁻¹	.12747×10⁻¹	62	.25184×10⁻²	.62322×10⁻²
22	.27156×10⁻¹	.62357×10⁻²	63	.21969×10⁻²	.66853×10⁻²
23	.26149×10⁻¹	.68824×10⁻²	64	.30328×10⁻²	.14594×10⁻¹
24	.24306×10⁻¹	.75654×10⁻²	65	.18754×10⁻²	.97370×10⁻²
25	.29653×10⁻¹	.74122×10⁻²	66	.82519×10⁻³	.86279×10⁻²
26	.22534×10⁻¹	.70915×10⁻²	67	.42867×10⁻³	.79559×10⁻²
27	.18558×10⁻¹	.61091×10⁻²	68	.26792×10⁻³	.82531×10⁻²
28	.17693×10⁻¹	.60876×10⁻²	69	.30950×10⁻¹	.15841×10⁻¹
29	.20294×10⁻¹	.71449×10⁻²	70	.42081×10⁻²	.87403×10⁻²
30	.16547×10⁻¹	.58783×10⁻²	71	.47404×10⁻²	.85657×10⁻²
31	.14025×10⁻¹	.56122×10⁻²	72	.53905×10⁻²	.85535×10⁻²
32	.12767×10⁻¹	.53557×10⁻²	73	.61050×10⁻²	.86556×10⁻²
33	.11956×10⁻¹	.52252×10⁻²	74	.68837×10⁻²	.87243×10⁻²
34	.11335×10⁻¹	.50757×10⁻²	75	.67873×10⁻²	.11733×10⁻¹
35	.33790×10⁻¹	.77028×10⁻²	76	.38437×10⁻²	.80086×10⁻²
36	.17929×10⁻¹	.60554×10⁻²	77	.35044×10⁻²	.79118×10⁻²
37	.38334×10⁻¹	.13278×10⁻¹	78	.24934×10⁻²	.96411×10⁻²
38	.22394×10⁻¹	.74177×10⁻²	79	.16932×10⁻²	.13428×10⁻¹
39	.23827×10⁻¹	.77759×10⁻²	80	.41795×10⁻³	.34892×10⁻¹
40	.23309×10⁻¹	.73372×10⁻²	81	.15111×10⁻²	.48004

[2] J. Kemeny, J. Snell, and G. Thompson, An introduction to finite mathematics, (Prentice-Hall, Inc., New York, N.Y., 1957).

The number $\bar{L}$ was computed from eq (3.5) and gave

$$\bar{L}=39.22 \tag{3.7}$$

as the average length of the game. The same result was obtained by summing the series in (3.6). The machine computing time was 12 min for the former calculation and 15 min for the latter.

4. Simulations

The random element in the game of Chutes and Ladders arises from the toss of a die at each move. This was simulated in our work by employing the pseudo-random number generator of Taussky and Todd [3]. That is, a sequence of pseudo-random numbers $t_1, t_2, \ldots$, where $0\leq t_i\leq 1$, were generated by the rules

$$\left.\begin{aligned} t_{j+1}&=2^{-35}r_j \\ r_{j+1}&=5^{15}r_j \pmod{2^{35}}, \quad r_0=1. \end{aligned}\right\} \tag{4.1}$$

The unit interval was divided into six equal subintervals and the result of the jth die throw was identified with the subinterval (first, second, . . ., sixth) in which t_j fell. The game was then played N times on the calculator. The total number of throws necessary to play these N games was recorded as $\sum_{i=1}^{N} L_i$, where L_i denotes the length of each game. The average length of the game was then computed as $(1/N)\sum_{i=1}^{N} L_i$. Furthermore, the fraction of these games which finished in 31 or less throws was computed. For $N=2^{14}$ the computing time was about 45 min. The results of these calculations are given in table 3.

TABLE 3. *Results of the Monte Carlo calculations*

N	$\frac{1}{N}\sum_{i=1}^{N} L_i$	Fraction of games finished by 31 throws
32	37.469	.56250
64	36.969	.57813
128	36.117	.55469
256	39.480	.46484
512	38.701	.49414
1024	39.521	.47168
2048	39.032	.49072
4096	39.288	.47974
8192	39.469	.47693
16384	39.420	.47406
Exact answer	39.224	.48004

5. Convergence

We first show how $(1/N)\sum_{i=1}^{N}\bar{L}_i$ should converge to $\bar{L}=39.22$. We introduce the distribution function

$$f(n)=X_{81}(n)-X_{81}(n-1). \tag{5.1}$$

[3] O. Taussky and J. Todd, Symposium on Monte Carlo methods (John Wiley & Sons, New York, N.Y., 1956).

The mean of $f(n)$ is simply the average length of the game $\overline{L}$ as given by eq (3.4). Furthermore, the standard deviation for the distribution function $f(n)$ is given by

$$\sigma=\sum_{n=1}^{\infty}(n-\overline{L})^2[X_{S1}(n)-X_{S1}(n-1)]. \qquad (5.2)$$

The value of σ for Chutes and Ladders was calculated according to (5.2) with $\overline{L}=39.224$ and the known values of $X_{S1}(n)-X_{S1}(n-1)$. We found $\sigma=25.222$.

Let $L_1, L_2, \ldots, L_N$ be regarded as a random sample of size N drawn from the population with distribution $f(n)$ where the moment generating function of $f(n)$ exists. It is well known [4] that even though $f(n)$ is not normally distributed, the "sample mean" $1/N\sum_{i=1}^{N}L_i$ has a population distribution which approaches a normal distribution with mean $\overline{L}$ and standard deviation $\sigma/\sqrt{N}$ as N becomes infinite. Hence the probability is approximately .95 that the average length of play computed from the first N simulations lies in an interval centered at $\overline{L}$ with length $4\sigma/\sqrt{N}$. This bound involves σ, which has been computed in the case of Chutes and Ladders.

In figure 1 the theoretical values of $\overline{L}$ and σ are used to plot the interval $(\overline{L}-2\sigma/\sqrt{N}, \overline{L}+2\sigma/\sqrt{N})$ as a function of N. The values of $(1/N)\sum_{i=1}^{N}L_i$ are also plotted and it is seen that they lie within our confidence interval.

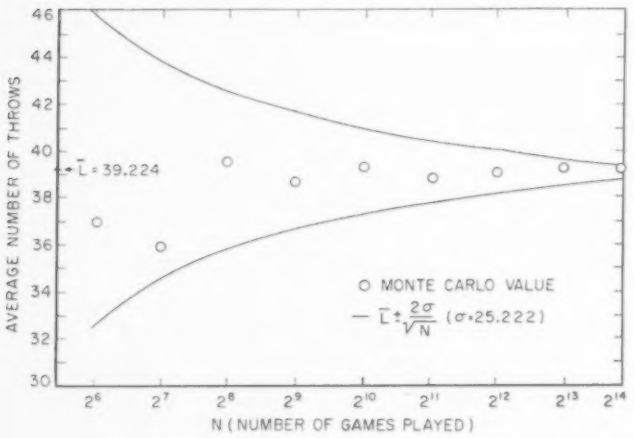
AVERAGE NUMBER OF THROWS
N (NUMBER OF GAMES PLAYED)
$\overline{L}$ = 39.224
O MONTE CARLO VALUE
$\overline{L} \pm \frac{2\sigma}{\sqrt{N}}$ (σ=25.222)

FIGURE 1. *Convergence to* $\overline{L}$ *by "*$N^{-1/2}$ *law."*

Now consider the probability of completing the game by a specified number of moves, say 31. If $p=X_{S1}(31)$, then the first N simulations can be regarded as N Bernoulli trials for which the probability of success in a single trial is p. The probability of exactly ν successes is given by the binomial distribution

$$\frac{N!}{\nu!(N-\nu)!}p^{\nu}(1-p)^{N-\nu} \qquad (p=X_{S1}(31), \quad 0\leq\nu\leq N). \qquad (5.3)$$

For large N, however, the fraction ν/N of successes is approximately normally distributed with mean p and standard deviation $\sqrt{p(1-p)/N}$. Approximately 95 percent of the area under this normal curve lies in the interval

$$p-\frac{2\sqrt{p(1-p)}}{\sqrt{N}}<X<p+\frac{2\sqrt{p(1-p)}}{\sqrt{N}}, \qquad (5.4)$$

and at least 95 percent lies in the larger interval

$$p-\frac{1}{\sqrt{N}}<X<p+\frac{1}{\sqrt{N}}. \qquad (5.5)$$

Thus p can be estimated from ν/N, as is illustrated in figure 2.

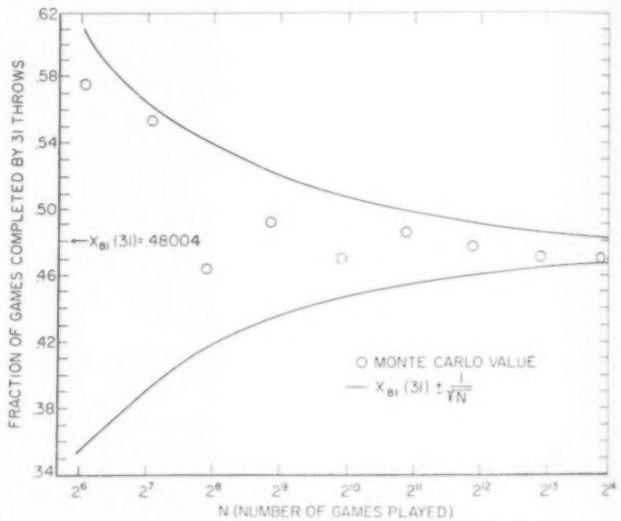

FIGURE 2. *Convergence to* $X_{S1}(31)$ *by "*$N^{-1/2}$ *law."*

6. Existence of the Moments of $f(n)$

In the previous section it was assumed that the moments $\sum_{n=1}^{\infty}n^k[X_{S1}(n)-X_{S1}(n-1)]$ exist for all $k\geq 1$. If we expand $X(n)$ in terms of the characteristic values of the matrix P (for example see Montroll,[5] p. 419) it follows that these moments will exist if P has $\lambda=1$ as the only eigenvalue with $|\lambda|=1$. However, our Markov chain has one absorbing state which can be reached from any other state. The probability of reaching this absorbing state is independent of the initial state; but, this is a necessary and sufficient condition (see Gantmacher,[6] p. 93) that

[4] W. Feller, An introduction to probability theory and its applications (John Wiley & Sons, New York, N.Y., 1957).

[5] E. Montroll, Markov chains, Weiner integrals, and quantum theory, Comm. Pure Appl. Math. 5, 419 (1952).

[6] F. Gantmacher, The theory of matrices, II (Chelsea Press, New York, N.Y., 1959).

there exist only one eigenvalue with magnitude 1. Thus, the moments exist; also, the average length of the game can be defined by eq (3.6).

7. Concluding Remarks

The Monte Carlo calculations are in agreement with the theoretical results obtained from the matrix calculations, and the convergence according to the "$N^{-1/2}$" laws is evident from figures 1 and 2. It is worth noting that the "$N^{-1/2}$" convergence implies that if an additional significant figure is desired in a Monte Carlo calculation the computing machine time requirements are increased by a factor of 100.

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Error Bounds in the Rayleigh-Ritz Approximation of Eigenvectors

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The difference between any eigenvector u_p of a linear operator A and its Rayleigh-Ritz approximation w_p is bounded in terms of the differences between the eigenvalues λ_i of A and their Rayleigh-Ritz upper bounds κ_i. The bound for the difference between u_p and w_p approaches zero with $\kappa_p-\lambda_p$.

1. Introduction

The most common method of approximating the eigenvalues $\lambda_1\leq\lambda_2\leq\ \ldots$ of a symmetric linear operator A is the Rayleigh-Ritz method [1,5,19].[1] This reduces an eigenvalue problem on a space of a large or even infinite number of dimensions to an eigenvalue problem on a space of relatively few dimensions.

If the desired eigenvalues λ_i are characterized as minima, the Rayleigh-Ritz approximations κ_i give upper bounds for them.

Along with the upper bounds κ_i for the eigenvalues the Rayleigh-Ritz method yields associated vectors w_i. It is to be expected that these vectors approximate the eigenvectors of A in some sense. Furthermore, it is to be expected that the better the eigenvalue λ_i is approximated by κ_i, the better will be the approximation of w_i to the corresponding eigenvector.

Indeed, this is easily seen in the case of the first eigenvector. If the unit vector w_1 is expanded in terms of the normalized eigenvectors u_i of A, we have

$$w_1=\sum a_i u_i, \tag{1.1}$$

$$\sum a_i^2=1, \tag{1.2}$$

and

$$\sum\lambda_i a_i^2=\kappa_1. \tag{1.3}$$

Subtracting λ_1 times (1.2) from (1.3) and noting that $\lambda_2\leq\lambda_3\leq\ \ldots$ we find that

$$a_1^2\geq\frac{\lambda_2-\kappa_1}{\lambda_2-\lambda_1} \tag{1.4}$$

or equivalently

$$\left[1-\frac{1}{2}(w_1-u_1,w_1-u_1)\right]^2\geq 1-\frac{\kappa_1-\lambda_1}{\lambda_2-\lambda_1}. \tag{1.5}$$

In general the λ_i are unknown, and we must express our results in terms of the κ_i and any lower bounds $\overline{\lambda}_i$ for the λ_i that may be available. Such bounds can be obtained by various methods (see for example [1,2,3,4,5,6,10,11,12,13,14,19,20,21,22,23,24,25,26]).

If $\lambda_i\geq\overline{\lambda}_i$, the inequality (1.5) leads to

$$\left[1-\frac{1}{2}(w_1-u_1,w_1-u_1)\right]^2\geq 1-\frac{\kappa_1-\overline{\lambda}_1}{\overline{\lambda}_2-\overline{\lambda}_1}. \tag{1.6}$$

This inequality shows that if the maximum error $\kappa_1-\overline{\lambda}_1$ is small compared with the interval $\overline{\lambda}_2-\overline{\lambda}_1$, the difference w_1-u_1 is small in norm. The bound (1.6) is sharp in the sense that equality is attained when $\lambda_i=\overline{\lambda}_i$ and $a_i=0$ for $i>2$. The inequality (1.6) is trivial for $\kappa_1\geq\overline{\lambda}_2$.

In this paper we generalize the bound (1.6). We give a bound for the norm of w_p-u_p in terms of the given bounds κ_i and $\overline{\lambda}_i$. This bound is again sharp in the sense that equality may be attained. The bound for (w_p-u_p,w_p-u_p) is small if the maximum error $\kappa_p-\overline{\lambda}_p$ is small relative to both $\overline{\lambda}_{p+1}-\kappa_p$ and $\overline{\lambda}_p-\kappa_{p-1}$. It becomes trivial if $\kappa_p\geq\overline{\lambda}_{p+1}$ or $\kappa_{p-1}\geq\overline{\lambda}_p$. The case $p=1$ gives an improved but more complicated version of (1.6).

If λ_p is multiple eigenvalue, we can only expect w_p to approximate one of the associated eigenvectors. Hence if λ_p lies near to several other eigenvalues we must expect w_p to approximate not u_p but a linear combination of the eigenvectors corresponding to the nearby eigenvalues. This approximation is established in section 3.

It is possible to find a bound for (w_p-u_p,w_p-u_p) by determining to what extent w_p satisfies the eigenvalue equation. Such bounds, which involve $(Aw_p-\kappa_p w_p,Aw_p-\kappa_p w_p)$ have been found by several authors [12, 20, 23]. Our bound, however, involves only the κ_i and the lower bounds $\overline{\lambda}_i$. It should be particularly useful in the case of differential operators where the Rayleigh-Ritz trial functions may not be sufficiently differentiable to give a finite value of (Aw_p,Aw_p). Our bounds are established by algebraic means for the case when A is an $N\times N$ matrix. They are independent of N. Consequent-

[1] Figures in brackets indicate the literature references at the end of this paper.

ly, the bounds also hold for infinite-dimensional operators A whose first p eigenvalues and eigenvectors are approximated uniformly by those of a sequence A_N of $N\times N$ matrices. This is certainly the case if A is completely continuous. It also holds under the weaker condition that A have p discrete eigenvalues defined by a minimum maximum principle. These must lie below any continuous spectrum. Thus A may be a Schroedinger operator corresponding to both bound and unbound states.

The fact that the eigenvalues λ_i are stationary values of the Rayleigh quotient tends to make the approximation of the eigenvectors worse than that of the eigenvalues. In fact, the bound (2.41) shows that the square of the norm of the error w_p-u_p is of the order $\kappa_p-\bar{\lambda}_p$.

The error bounds in sections 2 and 3 are in the sense of the norm. If A is a differential operator, its eigenvectors u_i are functions. It is often of interest to approximate the value of the function at a particular point. An adaptation of the method of Diaz and Greenberg [7, 9] which leads to such a pointwise approximation is presented in section 4.

2. Separated Eigenvalues

Let A be an hermitian $N\times N$ matrix. It is a linear operator on Euclidean N-space. Let the usual scalar product between two vectors u and v on this space be denoted by (u,v).

Let the eigenvalues of A be $\lambda_1\leq\lambda_2\leq\ \cdots\ \leq\lambda_N$. The corresponding eigenvectors are denoted by $u_1, \ldots, u_N$ so that the equations

$$Au_i=\lambda_i u_i \qquad i=1, \ldots, N \tag{2.1}$$

are satisfied. The u_i are normalized so that

$$(u_i,u_j)=\delta_{ij} \qquad i,j=1, \ldots, N. \tag{2.2}$$

Then (2.1) implies

$$(Au_i,u_j)=\lambda_i\delta_{ij} \qquad i,j=1, \ldots, N. \tag{2.3}$$

We suppose that the Rayleigh-Ritz method [1, 5, 19] is applied to find upper bounds

$$\kappa_\alpha\geq\lambda_\alpha \qquad \alpha=1, \ldots, M \tag{2.4}$$

for the first $M<N$ eigenvalues of A. This is done by choosing M linearly independent vectors $v_1, \ldots, v_M$ and finding the roots $\kappa_1\leq\kappa_2\leq\ \cdots\ \leq\kappa_M$ of the secular equation

$$\det\left[(Av_\alpha,v_\beta)-\kappa(v_\alpha,v_\beta)\right]=0 \qquad \alpha,\beta=1, \ldots, M. \tag{2.5}$$

Associated with each of the κ_α is a linear combination

$$w_\alpha=\sum_{\beta=1}^{M} C_\alpha^\beta v_\beta \tag{2.6}$$

of unit length such that the M linear equations

$$(Aw_\alpha,v_\beta)=\kappa_\alpha(w_\alpha,v_\beta) \qquad \beta=1, \ldots, M. \tag{2.7}$$

are satisfied. Then

$$(w_\alpha,w_\beta)=\delta_{\alpha\beta} \tag{2.8}$$

and

$$(Aw_\alpha,w_\beta)=\kappa_\alpha\delta_{\alpha\beta} \qquad \alpha,\beta=1, \ldots, M. \tag{2.9}$$

We assume that in addition to the upper bounds κ_α we have determined some lower bounds $\bar{\lambda}_i$ such that

$$\lambda_i\geq\bar{\lambda}_i \qquad i=1, \ldots, N. \tag{2.10}$$

This means that we have at least some idea of the degree of approximation of the κ_α to the eigenvalues λ_α.

We seek to determine from this degree of approximation of the κ_α to the eigenvalues the degree of approximation of a particular Rayleigh-Ritz eigenvector w_p to the eigenvector u_p of A.

We assume that $\lambda_{p-1}<\lambda_p<\lambda_{p+1}$ and that the bounds κ_{p-1}, κ_p, $\bar{\lambda}_p$, and $\bar{\lambda}_{p+1}$ are sufficiently good that

$$\kappa_p<\bar{\lambda}_{p+1} \tag{2.11}$$

and

$$\kappa_{p-1}<\bar{\lambda}_p. \tag{2.12}$$

Our problem is the following: Given the M vectors $w_1, \ldots, w_M$ satisfying (2.8), find the largest value of the deviation (w_p-u_p, w_p-u_p) of w_p from the pth normalized eigenvector of any matrix A satisfying the eq (2.9) and having eigenvalues λ_i satisfying (2.10).

If the eigenvalues λ_i of A are given, A is completely specified by prescribing its normalized eigenvectors. These eigenvectors $u_1, \ldots, u_N$ form a basis, so that the w_α can be written as linear combinations of them. Let

$$w_\alpha=\sum_{i=1}^{N} a_\alpha^i u_i \qquad \alpha=1, \ldots, M. \tag{2.13}$$

Since the u_i satisfy (2.2) and (2.3), the eqs (2.8) and (2.9) become

$$\sum_{i=1}^{N} a_\alpha^i a_\beta^i=\delta_{\alpha\beta} \tag{2.14}$$

and

$$\sum_{i=1}^{N} \lambda_i a_\alpha^i a_\beta^i=\kappa_\alpha\delta_{\alpha\beta} \qquad \alpha,\beta=1, \ldots, M. \tag{2.15}$$

On the other hand,

$$(w_p-u_p, w_p-u_p)=2(1-a_p^p), \tag{2.16}$$

If u_p satisfies (2.2) and (2.3), so does $-u_p$. We choose the sign to make a_p^p non-negative. Thus, our problem of maximizing (w_p-u_p, w_p-u_p) is reduced to that of minimizing $(a_p^p)^2$ under the constraints (2.14), (2.15), and (2.10).

We first keep the eigenvalues λ_i fixed and unequal. If a_α^i is the minimizing set of coefficients, we find by direct differentiation that the equations

$$a_p^p\delta_{ip}\delta_{\alpha p}+\sum_{\beta=1}^{M}(r_{\alpha\beta}+\lambda_i s_{\alpha\beta})a_\beta^i=0 \qquad \begin{matrix} i=1,\ldots,N \\ \alpha=1,\ldots,M \end{matrix} \tag{2.17}$$

must be satisfied. The $r_{\alpha\beta}$ and $s_{\alpha\beta}$ are Lagrange multipliers, and are symmetric:

$$r_{\alpha\beta}=r_{\beta\alpha}, \tag{2.18}$$

$$s_{\alpha\beta}=s_{\beta\alpha}.$$

To solve the eq (2.17), we multiply by a_γ^i and sum with respect to i. Using (2.14) and (2.15) we find that

$$a_p^p a_\gamma^p\delta_{\alpha p}+r_{\alpha\gamma}+\kappa_\gamma s_{\alpha\gamma}=0. \tag{2.19}$$

Interchanging α and γ and using (2.18), we find

$$r_{\alpha\gamma}=-(\kappa_\alpha-\kappa_\gamma)^{-1}\kappa_p a_p^p[a_\gamma^p\delta_{\alpha p}-a_\alpha^p\delta_{\gamma p}],$$

$$s_{\alpha\gamma}=(\kappa_\alpha-\kappa_\gamma)^{-1}a_\mu^p[a_\gamma^p\delta_{\alpha p}-a_\alpha^p\delta_{\gamma p}] \qquad \alpha\neq\gamma. \tag{2.20}$$

In particular, $r_{\alpha\gamma}$ and $s_{\alpha\gamma}$ vanish unless $\alpha=\gamma$ or either α or γ is p.

Letting $\alpha=\gamma$ in (2.19), we find

$$r_{\alpha\alpha}+\kappa_\alpha s_{\alpha\alpha}=-(a_p^p)^2\delta_{\alpha p}. \tag{2.21}$$

We substitute (2.20) and (2.21) in (2.17) to obtain

$$(\kappa_p-\kappa_\alpha)(\lambda_i-\kappa_\alpha)s_{\alpha\alpha}a_\alpha^i+(\lambda_i-\kappa_p)a_p^p a_\alpha^p a_p^i=0 \qquad \alpha\neq p \tag{2.22}$$

and

$$a_p^p\delta_{ip}+[(\lambda_i-\kappa_p)s_{pp}-(a_p^p)^2]a_p^i-\sum_{\beta\neq p}\frac{\lambda_i-\kappa_p}{\kappa_\beta-\kappa_p}a_p^p a_\beta^p a_\beta^i$$

$$=0, \qquad i=1,\ldots,N. \tag{2.23}$$

If the value $a_p^p=0$ is compatible with the constraints (2.14) and (2.15), the minimum value of $(a_p^p)^2$ is clearly zero. We suppose for the moment that this is not the case, so that

$$a_p^p\neq 0. \tag{2.24}$$

Then we can solve (2.23) with $i=p$ for s_{pp} in terms of the a_β^p. We can also eliminate $s_{\alpha\alpha}$ between (2.22) with $i=p$ and any other value of i to obtain

$$a_p^p a_\beta^p a_\beta^i=\frac{(\lambda_i-\kappa_p)(\lambda_p-\kappa_\beta)}{(\lambda_p-\kappa_p)(\lambda_i-\kappa_\beta)}a_p^i(a_\beta^p)^2. \tag{2.25}$$

Substituting this and the value of s_{pp} in (2.23) we obtain

$$a_p^i\left\{\frac{\lambda_i-\kappa_p}{\lambda_p-\kappa_p}\left[1-\sum_{\beta=1}^{M}(a_\beta^p)^2+\sum_{\beta=1}^{M}\frac{\lambda_p-\kappa_\beta}{\lambda_i-\kappa_\beta}(a_\beta^p)^2\right]\right\}$$

$$=0, \qquad i\neq p. \tag{2.26}$$

Suppose now that exactly L of the coefficients a_β^p are nonzero:

$$a_\beta^p\neq 0 \qquad \text{for} \qquad \beta=\beta_1,\beta_2,\ldots,\beta_L$$

$$\beta_1<\beta_2<\ldots<\beta_L. \tag{2.27}$$

By (2.24) one of the $\beta_\nu=p$. The term in braces in (2.26) times the product of the $(\lambda_i-\kappa_{\beta_\nu})$ with $\beta_\nu\neq p$ is a polynomial of degree L in λ_i. Hence it vanishes for at most L values of i. Consequently, $a_p^i\neq 0$ for $\overline{L}\leq L$ values of $i\neq p$.

From (2.22) it follows that $a_\nu^i=0$ implies $s_{\alpha\alpha}a_\alpha^i=0$ for all α. But $s_{\alpha\alpha}=0$ implies $a_\alpha^p=0$. Hence $a_p^i=0$ implies $a_{\beta_\nu}^i=0$. Thus our vectors w_α split into two subsets: The subset S_1 consists of L orthonormal vectors $w_{\beta_1},\ldots,w_{\beta_L}$ (including w_p) having only components in the $\overline{L}+1$ directions $u_{i_1},\ldots,u_{i_{\overline{L}+1}}$. The subset S_2 consists of $M-\overline{L}-1$ vectors orthogonal to u_p.

It follows that $\overline{L}$ must be either L or $L-1$. In the latter case, the L vectors w_{β_ν} in the first set will be the eigenvectors $u_{i_1},\ldots,u_{i_L}$ themselves. In particular, w_p is u_p so that the corresponding maximum of (w_p-u_p, w_p-u_p) is zero. This occurs if and only if $\kappa_p=\lambda_p$. The more interesting case is that in which

$$\overline{L}=L. \tag{2.28}$$

The term in the braces in (2.26) vanishes for $i=i_1,i_2,\ldots,i_{L+1}(i_1<i_2<\ldots<i_{L+1})$ except when $i_\nu=p$. This represents a set of L linear equations in the L unknowns $(a_{\beta_1}^p)^2\ldots,(a_{\beta_L}^p)^2$. It can be solved explicitly to yield

$$(a_{\beta_\mu}^p)^2=\prod_{\substack{\nu=1\\ \nu\neq\mu}}^{L}\frac{\lambda_p-\kappa_{\beta_\nu}}{\kappa_{\beta_\mu}-\kappa_{\beta_\nu}}\prod_{\substack{\tau=1\\ i_\tau\neq p}}^{L+1}\frac{\kappa_{\beta_\mu}-\lambda_{i_\tau}}{\lambda_p-\lambda_{i_\tau}} \tag{2.29}$$

(This result can be checked by the Lagrange interpolation formula [17]. A similar solution of a closely related equation was given by K. Loewner, Math. Z. *38*, 180–181 (1934).)

We now let $\beta=\beta_\mu$ in (2.25), multiply by a_ν^i, and sum. By (2.14) we have

$$\sum_{i_\tau\neq p}\frac{\lambda_{i_\tau}-\kappa_p}{\lambda_{i_\tau}-\kappa_{\beta_\mu}}(a_p^{i_\tau})^2+\frac{\lambda_p-\kappa_p}{\lambda_p-\kappa_{\beta\mu}}(a_p^p)^2=\delta_{p\beta_\mu} \qquad \mu=1,\ldots,L. \tag{2.30}$$

Again, we can solve explicitly for the L unknowns $(a_p^{i_\tau})^2, i_\tau\neq p$. Using (2.25) and (2.29), we find that

$$(a_{\beta_\sigma}^{i_\sigma})^2=\prod_{\substack{\nu=1\\ \nu\neq\mu}}^{L}\frac{\lambda_{i_\sigma}-\kappa_{\beta_\nu}}{\kappa_{\beta_\mu}-\kappa_{\beta_\nu}}\prod_{\substack{\tau=1\\ \tau\neq\sigma}}^{L+1}\frac{\kappa_{\beta_\mu}-\lambda_{i_\tau}}{\lambda_{i_\sigma}-\lambda_{i_\tau}} \tag{2.31}$$

The solution (2.31) formally satisfies the conditions (2.14), (2.15), and (2.17). (The square roots must be chosen so that a $\beta_{\beta_\mu}^{i_\sigma}$ is $(\lambda_{i_\sigma}-\kappa_{\beta_\mu})^{-1}$ times a function of i_σ only times a function of β_ν only.) In order to be admissible, the coefficients must be real. It is easily seen that this is the case if and only if

$$\lambda_{i_1}<\kappa_{\beta_1}<\lambda_{i_2}<\ \ldots\ <\kappa_{\beta_L}<\lambda_{i_{L+1}}. \tag{2.32}$$

The vectors w_β in the subset S_2 are orthogonal to $w_{\beta_1}, \ldots, w_{\beta_L}$ and u_p. It follows from (2.31) that they are orthogonal to $u_{i_1}, \ldots, u_{i_{L+1}}$. This is possible if and only if to each i with $i\neq i_1, \ldots, i_{L+1}$ there corresponds a separate κ_β with $\beta\neq\beta_1, \ldots, \beta_L$ such that $\kappa_\beta\geq\lambda_i$.

We now consider the possible minima of $(a_p^p)^2$. Choosing a particular set of β_ν and i_σ, we find from (2.31) that

$$(a_p^p)^2=\prod_{\beta_\nu\neq p}\frac{\lambda_p-\kappa_{\beta_\nu}}{\kappa_p-\kappa_{\beta_\nu}}\prod_{i_\tau\neq p}\frac{\kappa_p-\lambda_{i_\tau}}{\lambda_p-\lambda_{i_\tau}} \tag{2.33}$$

Because of (2.32)

$$\frac{(\lambda_p-\kappa_{\beta_\nu})(\kappa_p-\lambda_{i_\nu})}{(\kappa_p-\kappa_{\beta_\nu})(\lambda_p-\lambda_{i_\nu})}=1-\frac{(\kappa_p-\lambda_p)(\kappa_{\beta_\nu}-\lambda_{i_\nu})}{(\kappa_p-\kappa_{\beta_\nu})(\lambda_p-\lambda_{i_\nu})}<1 \tag{2.34}$$

for $i_\nu>p$ or $\beta_\nu<p$.

Thus, the right-hand side of (2.33) is increased by dropping any pair λ_{i_ν}, κ_{β_ν}. This means that its minimum will be attained when the sets $i_1, \ldots, i_{L+1}$, $\beta_1, \ldots, \beta_L$ are maximal with respect to the properties required of them. We further note that the right-hand side of (2.33) is an increasing function of λ_{i_τ} for $i_\tau\neq p$ and a decreasing function of κ_{β_ν} for $\beta_\nu\neq p$.

Keeping these facts in mind, we construct the sets $i_1, \ldots, i_{L+1}$, $\beta_1, \ldots, \beta_L$ which minimize $(a_p^p)^2$ for fixed unequal λ_i as follows.

Let

$$i_1=1. \tag{2.35}$$

Let

$$i_2=\min\{i|\exists\beta\ni\lambda_1<\kappa_\beta<\lambda_i\}. \tag{2.36}$$

Let

$$\beta_1=\max\{\beta|\kappa_\beta<\lambda_{i_2}\}. \tag{2.37}$$

Then inductively, let

$$i_{\nu+1}=\min\{i|\exists\kappa_\beta\ni\lambda_{i_\nu}<\kappa_\beta<\lambda_i\}, \tag{2.38}$$

$$\beta_\nu=\max\{\beta|\kappa_\beta<\lambda_{i_{\nu+1}}\}.$$

Because of (2.11) the set of β_ν includes p. If p is not included in the i_σ, we can easily construct a solution of the eqs (2.14) and (2.15) with $a_p^p=0$. To do this we define the $(a_{\beta_\mu}^{i_\sigma})^2$ by (2.31). Then $a_p^p=(w_p,u_p)=0$.

Conversely, if (2.24) is violated so that the minimum of $(a_p^p)^2$ is zero, we can consider the problem of minimizing a coefficient $(a_p^q)^2$ with $q<p$ under the constraints (2.14), (2.15), and $a_p^i=0$ for some set of $i<p$ including p. Since $\kappa_p<\lambda_{p+1}$, not all the a_p^q can vanish. Therefore this minimum problem will have a non-zero solution for some q and some set of $i\leq p$. The minimizing conditions again lead to the determination of sets i_σ and β_ν by (2.35), (2.36), (2.37), and (2.38). The integer p is included in the β_ν but not in the i_σ.

Thus, $a_p^p\neq0$ if and only if p is one of the i_σ. It follows from the construction of the i_σ that this will be the case if and only if there is a κ_β such that $\lambda_{p-1}<\kappa_\beta<\lambda_p$. This is assured by (2.12) for any λ_i satisfying (2.10). Condition (2.12) is therefore necessary and sufficient for $(a_p^p)^2$ to have a nonzero minimum.

The minimum value of $(a_p^p)_2$ is now given by (2.33). It is a continuous nondecreasing function of the eigenvalues λ_i. Hence its minimum with respect to the λ_i satisfying (2.10) will occur for $\lambda_i=\bar{\lambda}_i$. We may remove the assumption that the λ_i are unequal by a limiting process. This will alter the inequalities in (2.37) and (2.38) slightly.

As we pointed out in the introduction, we can replace the matrix A by a symmetric operator on a Hilbert space. We need only assume that A has at least p discrete eigenvalues defined by a minimum maximum principle [1,5,19]. For then the first N eigenvalues of A and their corresponding eigenvectors are uniformly approximated by those of an $N'\times N'$ matrix $A_{N'}$ for N' sufficiently large. If $\bar{\lambda}_i$ $i=1, \ldots, N$ are lower bounds for the first N eigenvalues of A, there is an $\epsilon_{N'}$ which goes to zero as $N'\to\infty$ such that $\bar{\lambda}_i-\epsilon_{N'}$ are lower bounds for the first N eigenvalues of $A_{N'}$. Also, $\bar{\lambda}_N-\epsilon_{N'}$ is a lower bound for the higher eigenvalues of $A_{N'}$. We let $N'\to\infty$ for fixed M and N. Using (2.33) and (2.34), we obtain the following theorem.

THEOREM 1. *Let $\kappa_1\leq\kappa_2\leq\ \ldots\ \leq\kappa_M$ be the Rayleigh-Ritz upper bounds for the first* M *of the eigenvalues $\lambda_1\leq\lambda_2\leq\ \ldots$ of a symmetric linear operator A. Let $\bar{\lambda}_1\leq\ \ldots\ \leq\bar{\lambda}_N$ be lower bounds for the first N eigenvalues of A, $N\geq M$.*

Let

$$\kappa_{p-1}<\bar{\lambda}_p<\kappa_p<\bar{\lambda}_{p+1}. \tag{2.39}$$

Define the numbers
$i_1<i_2<\ \ldots\ <i_{L+1}$, $\beta_1<\beta_2<\ \ldots\ <\beta_L$ by

$$i_1=1$$

$$i_2=\min\{i|\bar{\lambda}_i>\kappa_1\}$$

$$\beta_1=\max\{\beta|\kappa_\beta\leq\bar\lambda_{i_2}\}$$

$$i_{\nu+1}=\min\{i|\exists\beta_\nu\bar\lambda_{i_\nu}<\kappa_\beta<\bar\lambda_i\}$$

$$\beta_\nu=\max\{\beta|\kappa_\beta\leq\bar\lambda_{i_{\nu+1}}\} \tag{2.40}$$

Then if w_p is the normalized Rayleigh-Ritz eigenvector corresponding to the bound κ_p and u_p is the normalized eigenvector of A corresponding to the eigenvalue λ_p,

$$\left[1-\frac{1}{2}(w_p-u_p,w_p-u_p)\right]^2 \geq\left\{1-\frac{\kappa_p-\bar\lambda_p}{\bar\lambda_{i_{L+1}}-\bar\lambda_p}\right\}\prod_{\substack{\nu=1\\ \beta_\nu\neq p}}^{L}\left\{1-\frac{(\kappa_p-\bar\lambda_p)(\kappa_{\beta_\nu}-\bar\lambda_{i_\nu})}{(\kappa_p-\kappa_{\beta_\nu})(\bar\lambda_p-\bar\lambda_{i_\nu})}\right\} \tag{2.41}$$

The right-hand side of this inequality approaches zero if either κ_p approaches $\bar\lambda_{p+1}$ or κ_{p-1} approaches $\bar\lambda_p$. It is near one if the error $\kappa_p-\bar\lambda_p$ is small relative to the approximate spacing $\bar\lambda_{i_{L+1}}-\bar\lambda_p$ and if the products of errors $(\kappa_p-\bar\lambda_p)(\kappa_{\beta_\nu}-\bar\lambda_{i_\nu})$ are small relative to the products of approximate spacings $(\kappa_p-\kappa_{\beta_\nu})(\bar\lambda_p-\bar\lambda_{i_\nu})$.

If lower bounds $\bar\lambda_i$ are not given for all the eigenvalues λ_i, we can always use a lower bound for any particular eigenvalue as a lower bound for a higher eigenvalue. In particular, we can let $\bar\lambda_i=\bar\lambda_{p+1}$ for $i>p$ and $\bar\lambda_i=\bar\lambda_1$ for $i<p$. If $p>1$, (2.39) requires that $\kappa_{p-1}<\bar\lambda_p$. Then $L=2$, $i_1=1$, $\beta_1=p-1$, $i_2=\beta_2=p$, $i_3=p+1$. This leads to the simpler bound

$$\left[1-\frac{1}{2}(w_p-u_p,w_p-u_p)\right]^2 \geq\left\{1-\frac{\kappa_p-\bar\lambda_p}{\bar\lambda_{p+1}-\bar\lambda_p}\right\}\left\{1-\frac{(\kappa_p-\bar\lambda_p)(\kappa_{p-1}-\bar\lambda_1)}{(\kappa_p-\kappa_{p-1})(\bar\lambda_p-\bar\lambda_1)}\right\} \tag{2.42}$$

Even though this bound has fewer factors than (2.41), it is, in general, smaller.

For $p=1$ we can take $\bar\lambda_i=\bar\lambda_2$ for $i\geq2$. Then $L=1$, $i_1=\beta_1=1$, $i_2=2$, and (2.41) reduces to (1.6).

Example. We apply the Rayleigh-Ritz method to the matrix

$$A=\begin{pmatrix}1&1&0\\1&3&1\\0&1&23\end{pmatrix} \tag{2.43}$$

using trial vectors with vanishing third component. This amounts to finding the eigenvalues and eigenvectors of the 2×2 matrix obtained by striking out the third row and column of A. We obtain

$$\kappa_1=2-\sqrt{2},\ \kappa_2=2+\sqrt{2} \tag{2.44}$$

with the corresponding vectors

$$w_1\sim\left(\frac{1}{2}(2+\sqrt{2})^{1/2},\frac{1}{2}(2-\sqrt{2})^{1/2},0\right)$$

$$w_2\sim\left(\frac{1}{2}(2-\sqrt{2})^{1/2},\frac{1}{2}(2+\sqrt{2})^{1/2},0\right) \tag{2.45}$$

If the first two coordinate directions are replaced by the w_1 and w_2 directions, the matrix A becomes

$$A^*=\begin{pmatrix}\kappa_1&0&-\frac{1}{2}\kappa_1^{1/2}\\0&\kappa_2&-\frac{1}{2}\kappa_2^{1/2}\\-\frac{1}{2}\kappa_1^{1/2}&-\frac{1}{2}\kappa_2^{1/2}&23\end{pmatrix} \tag{2.46}$$

We now obtain lower bounds for the eigenvalues by means of Hadamard's theorem on determinants [16].

$$\bar\lambda_1=\kappa_1-\frac{1}{2}\kappa_1^{1/2},$$

$$\bar\lambda_2=\kappa_2-\frac{1}{2}\kappa_2^{1/2}. \tag{2.47}$$

For $\bar\lambda_3$ we use the fact that the largest eigenvalue must exceed the largest diagonal element. Then

$$\bar\lambda_3=23. \tag{2.48}$$

Substituting these values in (2.41) we find that

$$(w_1-u_1,w_1-u_1)\leq0.08025,$$
$$(w_2-u_2,w_2-u_2)\leq0.09980. \tag{2.49}$$

In this problem one can, of course, determine the eigenvalues and the corresponding eigenvectors explicitly. We find that

$$(w_1-u_1,w_1-u_1)=0.000328,$$
$$(w_2-u_2,w_2-u_2)=0.002238. \tag{2.50}$$

Thus, the error bounds (2.49) are a good deal larger than the errors themselves.

Our error bounds depend upon the lower bounds $\bar\lambda_i$. These were chosen rather crudely and could be improved in various ways (see, for example [5]). In order to determine the effect of such an improvement, we replace the $\bar\lambda_i$ by the eigenvalues λ_i in (2.41). We then obtain the bounds

$$(w_1-u_1,w_1-u_1)\leq0.000330,$$
$$(w_2-u_2,w_2-u_2)\leq0.002243. \tag{2.51}$$

These are very close to the actual values (2.50).

The simpler bound (1.6) gives

$$(w_1-u_1,\ w_1-u_1)\leq 0.195136 \tag{2.52}$$

if the values (2.47) of $\bar{\lambda}_i$ are used, and

$$(w_1-u_1,\ w_1-u_1)\leq 0.002364 \tag{2.53}$$

if they are replaced by the λ_i themselves. In both cases we see that the bound (2.41) is significantly better than the simplified bound (1.6).

3. Neighboring Eigenvalues

The condition (2.39) implies that the eigenvalue λ_p is simple. If this is not the case, the corresponding eigenvector u_p is not uniquely defined. In fact, if λ_p has multiplicity m, u_p may be any element of an m-space. If $m>1$, there will always be such a u_p orthogonal to w_p, so that the minimum of (w_p,u_p) is zero.

We must reformulate our problem. We seek the minimum value of $(w_p,u_p)^2$ when u_p is taken to be that element of the m-space which best approximates w_p. This u_p is the projection of w_p into the m-space of eigenvectors corresponding to λ_p.

The condition (2.39) implies not only that λ_p is simple, but that our bounds κ_{p-1}, κ_p, $\bar{\lambda}_p$, and $\bar{\lambda}_{p+1}$ are good enough to reveal its simplicity. That is, the error in our bounds is smaller than the separations between λ_{p-1}, λ_p, and λ_{p+1}. If this is not the case, we cannot distinguish between a simple and a multiple eigenvalue.

Suppose now that the upper and lower bounds for λ_p, λ_{p+1}, . . ., λ_{p+m-1} show these eigenvalues to lie close together. Suppose further that

$$\kappa_{p-1},\ <<\bar{\lambda}_p,\ \kappa_{p+m-1}<<\bar{\lambda}_{p+m} \tag{3.1}$$

so that λ_{p-1} and λ_{p+m} are known to lie away from the cluster of eigenvalues about λ_p. Then, to our degree of approximation λ_p behaves like an eigenvalue of multiplicity m. We ask how well w_p can be approximated by a linear combination of unit length of the eigenvectors u_p, u_{p+1}, . . ., u_{p+m-1}.

This problem is equivalent to that of minimizing $\sum_p^{p+m-1}(a_p^i)^2$ under the constraints (2.14), (2.15), and (2.10). By (2.20) we have $r_{\alpha\beta}=s_{\alpha\beta}=0$ unless $\alpha=\beta$ or α or $\beta=p$. Moreover, $r_{\alpha p}+\kappa_p s_{\alpha p}=0$, $r_{\alpha\alpha}+\kappa_\alpha s_{\alpha\alpha}=0$ for $\alpha\neq p$. This means that we would obtain the same minimizing conditions by imposing only the constraints (2.14) with $\alpha=\beta$ or α or $\beta=p$ and the single constraint (2.15) with $\alpha=\beta=p$. The latter may even be replaced by the inequality

$$\sum_{i=1}^{N}\lambda_i(a_p^i)^2\leq\kappa_p. \tag{3.2}$$

The other constraints (2.14) and (2.15) determine which local minima actually occur, but the local minima themselves are determined by (2.14) and (3.2).

The same situation applies in the case under consideration here. Necessary conditions for a maximum of $\sum_p^{p+m-1}(a_p^i)^2$ are determined by the constraints (2.14), (2.10), and (3.2).

Let $\tilde{u}_p$ be the unit vector in the direction of the projection of w_p into the space spanned by u_p, . . ., u_{p+m-1}:

$$\tilde{u}_p=\left\{\sum_{i=p}^{p+m-1}(a_p^i)^2\right\}^{-1/2}\sum_p^{p+m-1}a_p^iu_i. \tag{3.3}$$

Let $\tilde{u}_{p+1}$, . . ., $\tilde{u}_{p+m-1}$ be other linear combinations of u_p, . . ., u_{p+m-1} such that $\tilde{u}_p$, . . ., $\tilde{u}_{p+m-1}$ are orthonormal. Let $\tilde{u}_i=u_i$ for $i\neq p$, . . ., $p+m-1$ and put

$$\tilde{a}_\alpha^i=(w_\alpha,\ \tilde{u}_i). \tag{3.4}$$

Then by construction

$$\tilde{a}_p^i=0 \qquad i=p+1,\ .\ .\ .,\ p+m-1 \tag{3.5}$$

and

$$(\tilde{a}_p^p)^2=\sum_{i=p}^{p+m-1}(a_p^i)^2. \tag{3.6}$$

Moreover,

$$\sum_{i=1}^{N}\lambda_i(a_p^i)^2\geq\sum_1^{p-1}\lambda_i(\tilde{a}_p^i)^2+\lambda_p(\tilde{a}_p^p)^2+\sum_{p+m}^{N}\lambda_i(\tilde{a}_p^i)^2. \tag{3.7}$$

Thus, if we let

$$\tilde{\lambda}_i=\begin{cases}\lambda_p & i=p,\ .\ .\ .,\ p+m-1\\ \lambda_i & \text{otherwise.}\end{cases} \tag{3.8}$$

We have from (3.2) and (3.7) that

$$\sum_1^N\tilde{\lambda}_i(\tilde{a}_p^i)^2\leq\kappa_p. \tag{3.9}$$

Since the $\tilde{u}_i$ are orthonormal, (2.14) becomes

$$\sum\tilde{a}_\alpha^i\tilde{a}_\beta^i=\delta_{\alpha\beta}. \tag{3.10}$$

Our problem is thus reduced to minimizing $(\tilde{a}_p^p)^2$ under the constraints (3.10), (3.9), (2.10), and (3.5). The conditions for local minima are found to be as in section 2. However, the constraints (3.5) together with the fact that $\kappa_{p+m-1}<\bar{\lambda}_{p+m}$ relegate the vectors w_{p+1}, . . ., w_{p+m-1} to the set S_2 orthogonal to $\tilde{u}_p$. Furthermore, the conditions (3.5) eliminate the values $i=p+1$, . . ., $p+m-1$ from the i_α.

Thus we find the following theorem.

THEOREM 2. *Let $\kappa_1\leq\kappa_2\leq\ .\ .\ .\ \leq\kappa_M$ be the Rayleigh-Ritz upper bounds for the first M eigenvalues $\lambda_1\leq\lambda_2\leq\ .\ .\ .$ of an hermitian operator A. Let $\bar{\lambda}_1\leq\ .\ .\ .\ \leq\bar{\lambda}_N$ be lower bounds for the first $N\geq M$ eigenvalues. For a certain p and m let*

$$\kappa_{p-1} < \bar{\lambda}_p,$$

$$\kappa_{p+m-1} < \bar{\lambda}_{p+m} \tag{3.11}$$

Define the numbers $i_1, i_2, \ldots, i_{L+1}, \beta_1, \ldots, \beta_L$ *by* (2.40) *with* $\bar{\lambda}_{p+1}, \ldots, \bar{\lambda}_{p+m-1}$ *eliminated from the set of* $\bar{\lambda}_i$, *and* $\kappa_{p+1}, \ldots, \kappa_{p+m-1}$ *eliminated from the set of* κ_α.

Let w_p *be the Rayleigh-Ritz eigenvector that gives the bound* κ_p. *Then there exists a linear combination of unit length* $\tilde{u}_p$ *of the eigenvectors* $u_p, \ldots, u_{p+m-1}$ *of* A *such that*

$$\left[1-\frac{1}{2}(w_p-\tilde{u}_p, w_p-\tilde{u}_p)\right]^2 \geq \left\{1-\frac{\kappa_p-\bar{\lambda}_p}{\bar{\lambda}_{i_{L+1}}-\bar{\lambda}_p}\right\} \prod_{\substack{\nu=1 \\ \beta_\nu \neq p}}^{L} \left\{1-\frac{(\kappa_p-\bar{\lambda}_p)(\kappa_{\beta_\nu}-\bar{\lambda}_{i_\nu})}{(\kappa_p-\kappa_{\beta_\nu})(\bar{\lambda}_p-\bar{\lambda}_{i_\nu})}\right\} \tag{3.12}$$

As in section 2 we obtain a simpler but a weaker inequality by putting $\bar{\lambda}_i = \bar{\lambda}_{p+m}$ for $i > p+m$ and $\bar{\lambda}_i = \bar{\lambda}_1$ for $i < p$ when $p > 1$. This leads to

$$\left[1-\frac{1}{2}(w_p-\tilde{u}_p, w_p-\tilde{u}_p)\right]^2 \geq \left\{1-\frac{\kappa_p-\bar{\lambda}_p}{\bar{\lambda}_{p+m}-\bar{\lambda}_p}\right\} \left\{1-\frac{(\kappa_p-\bar{\lambda}_p)(\kappa_{p-1}-\bar{\lambda}_1)}{(\kappa_p-\kappa_{p-1})(\bar{\lambda}_p-\bar{\lambda}_1)}\right\}. \tag{3.13}$$

For $p=1$ we only have to put $\bar{\lambda}_i = \bar{\lambda}_{p+m}$ for $i > p+m$ to obtain

$$\left[1-\frac{1}{2}(w_1-\tilde{u}_1, w_1-\tilde{u}_1)\right]^2 \geq 1-\frac{\kappa_1-\bar{\lambda}_1}{\bar{\lambda}_{m+1}-\bar{\lambda}_1}. \tag{3.14}$$

By the same reasoning we can show that there is a linear combination $\tilde{u}_{p+q}$ of $u_p, \ldots, u_{p+m-1}$ that approximates w_{p+q} with $0 < q < m$. We eliminate $\kappa_p, \ldots, \kappa_{p+m-1}$ except for κ_{p+q} from the κ_α and $\bar{\lambda}_{p+1}, \ldots, \bar{\lambda}_{p+m-1}$ from the $\bar{\lambda}_i$ in forming the sets β_ν and i_σ. Then we obtain the inequalities (3.12) and (3.13) with w_p replaced by w_{p+q}, $\tilde{u}_p$ by $\tilde{u}_{p+q}$, and κ_p by κ_{p+q}.

EXAMPLE. We consider the vibrations of a uniform beam which is free at its ends and which lies on an elastic foundation with small linearly varying elastic constant. It satisfies the differential equation

$$u^{IV} + \epsilon x u = \lambda u \qquad 0 < x < 1 \tag{3.15}$$

with the end conditions

$$u''(0) = u'''(0) = u''(1) = u'''(1) = 0 \tag{3.16}$$

The constant ϵ is positive and small. We introduce the scalar product

$$(u, v) = \int_0^1 uv\,dx \tag{3.17}$$

and define the symmetric operator A by the bilinear form

$$(Au, v) = \int_0^1 [u''v'' + \epsilon uv]\,dx \tag{3.18}$$

If the eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots$ of A are defined as the successive minima of the Rayleigh quotient $(Av, v)/(v, v)$, they coincide with those of the problem (3.15), (3.16).

We are concerned with the two lowest eigenvalues. For $\epsilon = 0$ they both vanish. Corresponding eigenfunctions are 1 and $6^{-1/2}(1-2x)$. We use these as trial functions in the Rayleigh-Ritz method. We obtain

$$\kappa_1 = \frac{1}{2}(1-6^{-1/2})\epsilon,$$

$$\kappa_2 = \frac{1}{2}(1+6^{-1/2})\epsilon, \tag{3.19}$$

with the corresponding vectors

$$w_1 = 2^{-1/2}[1-6^{-1/2}(1-2x)]$$

$$w_2 = 2^{-1/2}[1+6^{-1/2}(1-2x)] \tag{3.20}$$

To obtain lower bounds $\bar{\lambda}_i$ we note that (Av, v) is greater for $\epsilon > 0$ than for $\epsilon = 0$. Thus, the λ_i are bounded below by the eigenvalues $\bar{\lambda}_i$ of the problem (3.15), (3.16) with $\epsilon = 0$. These can be found explicitly. We find

$$\bar{\lambda}_1 = \bar{\lambda}_2 = 0$$

$$\bar{\lambda}_3 = 500.462. \tag{3.21}$$

Condition (2.39) is violated so that we cannot say how well w_1 approximates u_1 without improving our bounds. However, we can use theorem 2 to state that there are linear combinations $\tilde{u}_1$ and $\tilde{u}_2$ of u_1 and u_2 such that

$$\int_0^1 (w_1-\tilde{u}_1)^2 dx \leq 2[1-\{1-0.0005912\epsilon\}^{1/2}],$$

$$\int_0^1 (w_2-\tilde{u}_2)^2 dx \leq 2[1-\{1-0.0014070\epsilon\}^{1/2}]. \tag{3.22}$$

Thus we have shown that w_1 and w_2 approximate linear combinations of u_1 and u_2 in the mean square sense.

When, as in this example, A is unbounded, it is often more desirable to have a bound for the deviation $(A(w_p-\tilde{u}_p), w_p-\tilde{u}_p)$ rather than $(w_p-\tilde{u}_p, w_p-\tilde{u}_p)$. In order to obtain such a bound we note that the quadratic form $(Av,v)+c(v,v)$ is positive definite for $c > -\bar{\lambda}_1$.

Hence we can define a new scalar product

$$[u,v]=(Au,v)+c(u,v). \tag{3.23}$$

We now define the operator $\hat{A}$ by

$$[\hat{A}u,v]=-(u,v). \tag{3.24}$$

Then the eigenvectors of $\hat{A}$ are multiples of those of A, and its eigenvalues are $-(\lambda_i+c)^{-1}$. Applying theorems 1 and 2 to A and expressing the results in terms of $\hat{A}$, we find the following.

THEOREM 3. *Under the hypotheses of theorem 2 there exists for any constant $c>-\bar{\lambda}_1$ a linear combination $\tilde{\tilde{u}}$ of $u_p, \ldots, u_{p+m-1}$ such that*

$$(A\tilde{\tilde{u}},\tilde{\tilde{u}})+c(\tilde{\tilde{u}},\tilde{\tilde{u}})=\kappa_p+c \tag{3.25}$$

and

$$\left[\kappa_p+c+\frac{1}{2}\{(Aw_p-A\tilde{\tilde{u}},w_p-\tilde{\tilde{u}})+c(w_p-\tilde{\tilde{u}},w_p-\tilde{\tilde{u}})\}\right]^2 \geq (\kappa_p+c)(\bar{\lambda}_p+c)\left\{1-\frac{\kappa_p-\bar{\lambda}_p}{\bar{\lambda}_{i_{L+1}}-\bar{\lambda}_p}\right\}\prod_{\substack{\nu=1\\ \beta_\nu\neq p}}^{L}\left\{1-\frac{(\kappa_p-\bar{\lambda}_p)(\kappa_{\beta_\nu}-\bar{\lambda}_{i_\nu})}{(\kappa_p-\kappa_{\beta_\nu})(\bar{\lambda}_p-\bar{\lambda}_{i_\nu})}\right\} \tag{3.26}$$

When the multiplicity $m=1$, $\tilde{\tilde{u}}$ is a multiple of u_p and we have the analog of theorem 1.

Applying theorem 3 to the example (3.15), (3.16), we obtain the inequality

$$\int_0^{\prime} \tilde{u}''^2+(c+\epsilon x)(w_1-\tilde{\tilde{u}})^2]dx \leq 2[c+1.704\epsilon-c^{1/2}\{c+1.704\epsilon\}^{1/2}\{1-0.0005912\epsilon\}^{1/2}] \tag{3.27}$$

for any $c\geq 0$. (The function $\tilde{\tilde{u}}$ depends upon c, however.)

4. Pointwise Bounds for Eigenfunctions

When A is a differential operator, theorems 1 and 3 give bounds for the mean square deviation of the approximate eigenfunction w_p from the exact eigenfunction u_p. It is often of interest to determine the value of u_p at a particular point.

In certain cases a pointwise bound for the deviation $|w_p-\tilde{\tilde{u}}|$ at a point comes directly from the bound (3.26) of theorem 3. For example, we show that for any $0\leq\zeta\leq 1$

$$|w_1(\xi)-\tilde{\tilde{u}}(\xi)|^2\leq G(\xi,\xi)\int_0^1[(w_1-\tilde{\tilde{u}})''^2+c(w_1-\tilde{\tilde{u}})^2]dx, \tag{4.1}$$

where

$$G(\xi,\xi)=\frac{1}{8}b^{-3}[\sinh^2 b-\sin^2 b]^{-1}$$

$$\times[\sinh b\cosh b-\sin b\cos b+2\sinh b\cosh b(1-2\xi)-2\sin b\cos b(1-2\xi)+(\sinh b\cos b+\sin b\cosh b)\sin b(1-2\xi)\sinh b(1-2\xi)+(\sinh b\cos b-\sin b\cosh b)\cos b(1-2\xi)\cosh b(1-2\xi)]$$

$$b=\left(\frac{1}{4}c\right)^{1/4} \tag{4.2}$$

(See, for example, [8]). Thus, (3.27) gives a bound for $|w_1(\xi)-\tilde{\tilde{u}}(\xi)|$.

In the case of partial differential operators such a bound may or may not exist. If it exists, it is difficult to find.

However, one can use the following adaptation of the method of Diaz and Greenberg [7,9])cf. [15,18]). For the sake of simplicity we present it only for the case of a special second order operator in two dimensions.

Consider the eigenvalue problem

$$-\Delta u+r(x,y)u=\lambda q(x,y)u \tag{4.3}$$

on a two-dimensional domain D with smooth boundary C. Here Δ is the usual Laplace operator. The function q is positive and r is non-negative, and both are continuous in the closure of D. The boundary C consist of two parts C_1 and C_2, and we have boundary conditions

$$u=0 \quad \text{on} \quad C_1$$

$$\frac{\partial u}{\partial n}+k(x,y)u=0 \quad \text{on} \quad C_2, \qquad k\geq 0. \tag{4.4}$$

We define the scalar product

$$(u,v)=\iint_D quvdxdy \tag{4.5}$$

on the linear vector space of functions which are piecewise continuously differentiable in D and vanish on C_1.

Let u_p be the normalized eigenfunction corresponding to the eigenvalue λ_p, and let w_p be the function corresponding to a Rayleigh-Ritz upper bound κ_p for λ_p. Theorem 1 gives a bound for the deviation in norm (w_p-u_p, w_p-u_p).

We wish to approximate the value of u_p at an interior point of D, which we choose as the origin of our coordinate system. We use the fact that u_p satisfies the differential equation (4.3) with $\lambda=\lambda_p$ and the boundary conditions (4.4). Let $\Gamma(x, y)$ be a parametrix for the differential equation (4.3) satisfying (4.4). That is,

$$\Gamma(x,y)=-(4\pi)^{-1}\log(x^2+y^2)+\phi(x,y), \tag{4.6}$$

where ϕ is any twice continuously differentiable function such that

$$\Gamma=0 \qquad \text{on} \quad C_1$$

$$\frac{\partial\Gamma}{\partial n}+k\Gamma=0 \qquad \text{on} \quad C_2 \tag{4.7}$$

Multiplying (4.3) by Γ and integrating by parts, we find that

$$u_p(0,\ 0)=\iint_D u_p[-\Delta\phi+(r-\lambda_p q)\Gamma]dxdy. \tag{4.8}$$

Replacing u_p by w_p and λ_p by κ_p on the right, we obtain the value

$$w_p^*(0,\ 0)=\iint_D w_p[-\Delta\phi+(r-\kappa_p q)\Gamma]dxdy \tag{4.9}$$

which can be computed by quadratures. Using Schwarz's inequality, the normalization of u_p, and the triangle inequality, we find

$$|w_p^*(0,\ 0)-u_p(0,\ 0)|\leq(\kappa_p-\lambda_p)(\Gamma,\ \Gamma)^{1/2}$$

$$+(w_p-u_p,\ w_p-u_p)^{1/2}$$

$$\left\{\iint_D q^{-1}[-\Delta\phi+(r-\kappa_p q)\Gamma]^2dxdy\right\}^{1/2}. \tag{4.10}$$

Thus, the bound (2.41) for $(w_p-u_p,\ w_p-u_p)$ together with the bound $\kappa_p-\overline{\lambda}_p$ for $\kappa_p-\lambda_p$ provides explicit upper and lower bounds for $u_p(0,\ 0)$. These bounds lie close together if the error bounds (2.41) and $\kappa_p-\overline{\lambda}_p$ are small.

The same method applies to the function $\tilde{u}_p$ of theorem 2. If

$$\tilde{u}_p=\sum_p^{p+m-1} a^i u_i, \tag{4.11}$$

we find from (4.8) that

$$\tilde{u}_p(0,\ 0)=\sum a^i\iint_D u_i[-\Delta\phi+(r-\lambda_i q)\Gamma]dxdy. \tag{4.12}$$

Hence if we again define the approximate value w_p^* by (4.9), we obtain

$$|w_p^*(0,\ 0)-\tilde{u}_p(0,\ 0)|\leq\ \max\{\kappa_p-\overline{\lambda}_p,\ \kappa_{p+m-1}-\kappa_p\}$$

$$(\Gamma,\ \Gamma)^{1/2}+(w_p-\tilde{u}_p,\ w_p-\tilde{u}_p)^{1/2}$$

$$\left\{\iint_D q^{-1}[-\Delta\phi+(r-\kappa_p q)\Gamma]^2dxdy\right\}^{1/2}. \tag{4.13}$$

This inequality together with theorem 2 gives upper and lower bounds for $\tilde{u}_p(0,\ 0)$.

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Sequence Transformations Based on Tchebycheff Approximations

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Let $P=\{p_n\}$ be a real sequence and define the transformed sequence $\mathcal{C}(P)$ as follows: Consider the segment, p_m, p_{m+1}, p_{m+2}, p_{m+3} of P and determine a_m, b_m, and c_m so that $\max_n|a_m b_m^n+c_m-p_n|$, $n=m,\ m+1,\ m+2,\ m+3$ is minimized. The mth term of $\mathcal{C}(P)$ is taken to be c_m. The effect of the transformation $\mathcal{C}$ on sequences of the type

$$p_0+\sum_{i=1}^{k}\alpha_i\lambda_i^n,\quad p_0+(-1)^n\sum_{i=1}^{k}a_i n^i/\sum_{i=1}^{l}b_i n^i,\quad \text{and}\quad p_0+\frac{a\lambda^n}{n}$$

is considered. In each case $\mathcal{C}$ is shown to be very effective in accelerating convergence or decelerating divergence. For example, if the second sequence behaves as an^k as $n\to\infty$, then the transformed sequence behaves as an^{k-3}. A similar transformation $\mathcal{D}$ is defined by approximating in the Tchebycheff sense a segment of P by $ab_n\cos(\theta+n\phi)+c$. The effect of $\mathcal{D}$ is studied for sequences of the above type and also for $p_0+\Sigma_{i=1}^{k}\alpha_i\lambda_i^n\cos(\theta_i+n\phi_i)$. These sequence transformations are similar in nature to Aitken's δ^2-process and its generalization. A comparison of the two types of transformations is made. Several examples are given to illustrate the effect of $\mathcal{C}$ and $\mathcal{D}$ on various sequences.

1. Introduction

Let $P=\{p_n|n=1,2,\ \ldots\}$ be a real sequence and let $\mathcal{T}$ be a sequence transformation with

$$Q=\mathcal{T}(P)=\{t_n(p_1,p_2,\ \ldots,\ p_n)|n=1,\ 2,\ \ldots\}.$$

A large variety of such transformations have been studied, many in the modified form of summability methods for infinite series, which are designed to accelerate or induce the convergence of the sequence P. Such transformations may be based on approximating a segment of P by a function of a given form. Assume that p_n behaves approximately as $f(A,n)$ where A represents some parameters to be determined. Consider a segment $\{p_i|i=1,2,\ \ldots,\ m\}$ of P. The parameters A_m may be determined so as to minimize the error of the approximation in some sense. Since $f(A_m,n)$ is a known function of n we may set

$$t_m(p_1,p_2,\ \ldots,\ p_m)=\lim_{n\to\infty} f(A_m,n).$$

The transformations of this paper result from assuming p_n to behave as ab^n+c or $ab^n\cos(\theta+n\phi)+c$. The transformations are denoted by $\mathcal{C}$ and $\mathcal{D}$, respectively. The approximation to give the mth term of $\mathcal{C}(P)$ and $\mathcal{D}(P)$ is based on four or six, respectively, terms from P. The parameters are determined so as to minimize

$$\max_n|a_m b_m^n+c_m-p_n|,\qquad n=m,\ m+1,\ m+2,\ m+3$$

or

$$\max_n|a_m b_m^n\cos(\theta_m+n\varphi_m)+c_m-p_n|,\qquad n=m,\ \ldots,\ m+5$$

If $b_m<1$, then $\lim_{n\to\infty} f(A_m,n)$ is just c_m; if $b_m>1$ then this limit does not exist, but the value c_m is still taken, c_m being the *antilimit*.

Explicit formulas for c_m in terms of the p_i have been obtained in [5].[2] These formulas are not linear in the p_i and the transformations exhibit some marked differences from the classical linear transformations. In many cases $\mathcal{C}$ and $\mathcal{D}$ are much more powerful and in other cases P may converge quite rapidly and the transformed sequence may be identically infinite. Although $\mathcal{C}$ and $\mathcal{D}$ are not regular they do have a property of *joint convergence*, namely that if both P and the transformed sequences converge then they have the same limit.

It is not surprising that $\mathcal{C}$ and $\mathcal{D}$ are effective for sequences of exponential type

$$P=\left\{p_0+\sum_{i=1}^{k}\alpha_i\lambda_i^n|n=1,\ 2,\ \ldots\right\}. \tag{1}$$

If $|\lambda_i|>|\lambda_{i+1}|$ then $\mathcal{C}(P)$ converges or diverges as $(\lambda_2)^n$ instead of $(\lambda_1)^n$. $\mathcal{D}(P)$ behaves as $(\lambda_3)^n$ or $(\lambda_2^2/\lambda_1)^n$. $\mathcal{D}$ is effective for sequences of the more general type

$$P=\left\{p_0+\sum_{i=1}^{k}\alpha_i\lambda_i^n\cos(\theta_i+n\phi_i)|n=1,\ 2,\ \ldots\right\}. \tag{2}$$

and C is useful for such sequences in some cases.

These transformations are also very powerful for alternating rational sequences:

$$P=\left\{p_0+(-1)^n\sum_{i=1}^{k_1}a_i n^i/\sum_{i=1}^{k_2}b_i n^i|n=1,\ 2,\ \ldots\right\}. \tag{3}$$

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[2] Figures in brackets indicate the literature references at the end of this paper.

It is shown that $\mathcal{C}$ accelerates the convergence by a factor of $1/n^3$. That $\mathcal{D}$ accelerates the convergence even more can be seen from the examples although the actual factor of acceleration has not been determined. An example shows that $\mathcal{C}$ and $\mathcal{D}$ are very effective for a sequence of the form

$$P=\left\{\frac{(-1)^n\lambda^n}{n}\ \middle|\ n=1, 2, \ldots\right\}. \tag{4}$$

The basic principle of these transformations is the same as that of Aitken's δ^2-process and its generalizations. $\mathcal{C}$ is the direct analog of the δ^2-process. Shanks [8] has given the most complete account of this theory. $\mathcal{C}$ and $\mathcal{D}$ have many properties in common with these transformations. The analogs of $\mathcal{C}$ and $\mathcal{D}$ in Shanks' work have the same power as $\mathcal{C}$ and $\mathcal{D}$ for sequences of type (1) and (2). For sequences of type (3), $\mathcal{C}$ and $\mathcal{D}$ are definitely more powerful. Aitken's δ^2-process accelerates the convergence of (3) by a factor of only $1/n^2$. $\mathcal{C}$ and $\mathcal{D}$ are much more effective for the sequence (4). There are sequences for which the δ^2-process and its generalizations are more effective, but as a general guide $\mathcal{C}$ and $\mathcal{D}$ are as effective or more effective than their analogs in Shanks' paper.

A comparison of this paper with Shanks' will indicate that $\mathcal{C}$ and $\mathcal{D}$ have many properties which have not been developed. The main purpose here has been to establish these transformations as tools of numerical analysis and not of analysis. When reading Shanks' paper one has the feeling that much remains to be done in the study of these transformations and that a complete understanding of their behavior for sequences of real and complex functions will require a penetrating analysis.

It is typical but somewhat disappointing that these transformations may be most effective for easy problems and least effective for difficult problems. If one is solving three simultaneous linear equations by Gauss-Seidel iteration and the characteristic values determining the rate of convergence are .95, .5, and .1, then these transformations increase the rate of convergence dramatically. But if one has 100 equations with 10 characteristic values between .99 and .995, then the elimination of even the five largest ones is not very significant. On the other hand if the largest characteristic value is .995 and the others are, say, less than .9 the transformations are very effective in accelerating the convergence.

2. Derivation of the Transformations

2.1. Preliminaries

Sequences are denoted by P, Q, . . . and their elements by p_n, q_n, ... A sequence transformation is denoted by a script letter as $\mathcal{C}$ and $\mathcal{D}$. Braces, $\{\ \}$, denote a sequence or set and $\{x|\ \ldots\}$ is read "the set of x such that . . ." We denote by $O(x)$ and $o(x)$ two functions such that

$$\lim_{x\to\infty}\frac{o(x)}{x}=0$$

and, for x sufficiently large,

$$\left|\frac{O(x)}{x}\right|\leq \text{constant}.$$

Let P be a sequence of real numbers. A transformed sequence $Q=\{q_n|n=1, 2, \ldots\}$ of P, is obtained as follows: Take a segment, $\{p_i|i=m, \ldots, m+k\}$ of P, and approximate the values p_n as a function of n by ab_n+c or $ab_n\cos(\theta+n\phi)+c$. The corresponding c value is then assigned to q_m.

For these particular transformations to be effective, the sequence P must, in some sense, behave exponentially. As an example of such sequences consider the real k-vectors defined by

$$v_{n+1}=Av_n,$$

where A is a real $k\times k$ matrix. Let v^* be the solution of

$$(A-I)v^*=0.$$

Then $v_{n+1}-v^*=A_n(v_0-v^*)$. The sequence of vectors defined will converge to v^* if $A^n(v_0-v^*)$ tends to zero as n tends to infinity. If A is a normal matrix with characteristic vectors u_i and distinct real characteristic values λ_i, then

$$A^n(v_0-v^*)=\sum_{i=1}^{k}\alpha_i\lambda_i^n u_i.$$

If the λ_ir are complex then

$$A^n(v_0-v^*)=\sum\alpha_i|\lambda_i|^n\cos\ (\theta_i+n\phi_i)u_i.$$

2.2. Tchebycheff Approximations

We wish to approximate a segment of P by ab^n+c. The use of least squares approximation appears to be impossible due to the difficulty of the nonlinear equations involved. In [5] the theory of approximation by ab^x+c in the Tchebycheff sense is developed in detail. The following result is given. Let p_m, p_{m+1}, p_{m+2}, and p_{m+3} be four consecutive values of P to be approximated. Then c_m of the best approximation, $a_m(b_m)^n+c_m$, is given by

$$c_m=\frac{(p_m+p_{m+1})(p_{m+2}+p_{m+3})-(p_{m+1}+p_{m+2})^2}{2(p_m-p_{m+1}-p_{m+2}+p_{m+3})}. \tag{5}$$

Let $p_m, \ldots, p_{m+5}$ be consecutive values from P to be approximated by $ab^n\cos(\theta+n\phi)+d$. The theory of Tchebycheff approximation by $ab^x\cos(\theta+x\phi)+d$ is not complete and an explicit formula for d_m of the best approximation has not been rigorously established. However it is conjectured that the following formula is valid. Set $s_i=\frac{1}{2}(p_{m+i}+p_{m+i+1})$, $i=0, 1, \ldots, 4$. Then

$$d_m=\frac{2s_1s_2s_3+s_4(s_0s_2-s_1^2)-s_0s_3^2-s_2^3}{2s_2(s_0+2s_1-3s_2+2s_3+s_4)-2(s_1-s_3)^2-4(s_1s_4+s_0s_3)+2s_0s_4}. \tag{6}$$

This formula is based on the assumption that the best Tchebycheff approximation is characterized by the alternation five times of the error function. See [6] for a discussion of the characterization of best Tchebycheff approximations.

Note that (6) may be written in a much simpler form. Let $s_i'=(s_i-s_2)$, $i=0,1,3,4$; then

$$d_m=s_2-\frac{s_4'(s_1')^2+s_0'(s_3')^2}{s_0's_4'-(s_1'-s_3')^2-2(s_1's_4'+s_3's_0')}.$$

2.3. The Transformations $\mathcal{C}$ and $\mathcal{D}$

Let P be given, then a new sequence Q is found as follows: For each m such that the denominator of (5) is nonzero $q_m=c_m$; if the denominator of (5) is zero and the numerator does not vanish then we assign $q_m=\infty$; if both the numerator and denominator are zero $q_m=p_m$. This transformation of P into Q may be written in operator form as

$$Q=\mathcal{C}(P).$$

A transformation $\mathcal{D}$ is defined in a similar manner by (6).

The repeated application k times of $\mathcal{C}$ and $\mathcal{D}$ is denoted by $\mathcal{C}^k(P)$ and $\mathcal{D}^k(P)$.

3. Analysis of the Transformations

3.1. Algebraic Properties of $\mathcal{C}$

The study of the properties of these transformations begins with a simple algebraic property of $\mathcal{C}$.

If $P=\{p_n\}$ then $aP+b$ is defined as $\{ap_n+b|n=1,2,\ldots\}$. The following result may be established by direct computation from (5).

THEOREM 1: $\mathcal{C}(aP+b)=a\mathcal{C}(P)+b.$

It is not true that $\mathcal{C}(P_1+P_2)=\mathcal{C}(P_1)+\mathcal{C}(P_2)$.

3.2. Transformation of Exponential Sequences by $\mathcal{C}$

Let P be a real sequence of the form

$$\left\{p_n=p_0+\sum_{i=1}^{k}\alpha_i\lambda_i^n|n=1,2,\ldots\right\}$$

with $|\lambda_i|>|\lambda_{i+1}|$. If $|\lambda_1|>1$, P does not converge but *diverges from* p_0. In such a case p_0 is the *antilimit* of P. This sequence is of an exponential type and $\mathcal{C}$ should be effective in increasing the rate of convergence or decreasing the rate of divergence.

Let $P'=P-p_0$; then

$$\mathcal{C}(P)=\mathcal{C}(P')+p_0.$$

The general term of $\mathcal{C}(P')$ may be explicitly computed to be

$$\frac{\sum_{i<j}\alpha_i\alpha_j\lambda_i^n(\lambda_j/\lambda_1)^n(1+\lambda_i)(1+\lambda_j)(\lambda_i-\lambda_j)^2}{\alpha_1(1-\lambda_1-\lambda_1^2+\lambda_1^3)+\sum_{i=2}^{k}\alpha_i(\lambda_i/\lambda_1)^n(1-\lambda_i-\lambda_i^2+\lambda_i^3)}.$$

The denominator of this expression may have only a finite number of zeros as a function of n. We may write the denominator as

$$\alpha_1(1-\lambda_1-\lambda_1^2+\lambda_1^3)+o(1)$$

since $|\lambda_j/\lambda_1|<1$ for $j>1$.

The largest term in the numerator is λ_2^n. Others among the larger terms are

$$\lambda_3^n,\quad(\lambda_2\lambda_3/\lambda_1)^n.$$

Thus $\mathcal{C}$ is seen to eliminate the largest exponential term from P'.

The repeated application of $\mathcal{C}$ will eliminate the largest remaining term. The largest term in $\mathcal{C}^2(P')$ is λ_3^n.

The above analysis has established:

THEOREM 2: *Let* $P=\{p_0+\sum_{i=1}^{k}\alpha_i\lambda_i^n\}$, $|\lambda_i|>|\lambda_{i+1}|$; *then*

$$\mathcal{C}(P)=\left\{p_0+\frac{\lambda_2^n\sum_{i=0}^{k'}d_i(\mu_i)^n}{1+o(1)}\right\}\qquad\begin{matrix}\mu_0=1\\ |\mu_i|<1,\quad i\geq 1.\end{matrix}$$

Thus $\mathcal{C}$ is seen to have a desirable effect on sequences of this type, which was to be expected. In the next section it is seen that $\mathcal{C}$ is effective for some sequences of a completely different nature.

3.3. Transformations of Alternating Rational Sequences by $\mathcal{C}$

Let P be a sequence of the form

$$\left\{p_n=p_0+(-1)^n\sum_{i=0}^{k_1}a_in^i\Big/\sum_{i=0}^{k_2}b_in^i\right\}.$$

We shall consider $\mathcal{C}(P)$.

First take the special case

$$P=\{p_0+(-1)^nan^k\}.$$

The nth term of $\mathcal{C}(P)$ is

$$p_0+a\frac{(n^k-(n+1)^k)((n+2)^k-(n+3)^k)-((n+1)^k-(n+2)^k)^2}{2(-1)^n(n^k+(n+1)^k-(n+2)^k-(n+3)^k)}.$$

After some manipulation this term is seen to be of the form

$$p_0+\frac{an^{2k-4}k^2(k-1)}{8(-1)^n kn^{k-1}}+O(n^{k-4})$$

or

$$p_0+a(-1)^n n^k\left[\frac{k(k-1)}{8n^3}+O\left(\frac{1}{n^4}\right)\right].$$

Now consider the general situation. p_n may be written as

$$p_n=p_o+\alpha_0' n^k+\sum_{i=1}^{\infty}\alpha_1' n^{k-i} \qquad k=k_1-k_2$$

for n sufficiently large. It is clear that the denominator of c_m is $8(-1)^n kn^{k-1}+O(n^{k-2})$ since it is linear in the terms of P. The denominator of c_m has only a finite number of zeros as a function of m. The terms in the numerator are of the form

$$\alpha_i'\alpha_j'[n^{k-i}-(n+1)^{k-i}][(n+2)^{k-j}-(n+3)^{k-j}]$$

$$+\alpha_i'\alpha_j'[n^{k-j}-(n+1)^{k-j}][(n+2)^{k-i}-(n+3)^{k-i}]$$

$$-2\alpha_i'\alpha_j'[(n+1)^{k-i}-(n+2)^{k-i}][(n+1)^{k-j}-(n+2)^{k-j}].$$

By direct calculation it is seen that this term is $O(n^{2k-i-j-4})$.

Thus we have established:

THEOREM 3: *Let* $P=\{p_0+(-1)^n\sum_{i=0}^{k_1}a_i n^i/\sum_{i=0}^{k_2}b_i n^i\}$; *then for* m *sufficiently large the* m*th term of* $\mathcal{C}(P)$ *is*

$$p_0+\sum_{i=1}^{\infty}\alpha_i'(-1)^m m^{k-i-3}$$

where $k=k_1-k_2$, $\alpha_0'=a_{k_1}k(k-1)/8b_{k_2}$.

Examples of sequences which are approximately of this type are given by the partial sums of alternating infinite series of rational functions of n. One can consider such sequences directly and establish an analog of theorem 3.

3.4. Regularity and Joint Convergence

A transformation $\mathcal{T}$ is said to be regular if the existence of $\lim_{n\to\infty} p_n$ implies $\lim_{n\to\infty} q_n=\lim_{n\to\infty} p_n$ where $Q=\mathcal{T}(P)$. Since $\mathcal{C}$ is a transformation designed to accelerate or induce convergence one would like for $\mathcal{C}$ to be regular. That $\mathcal{C}$ is not regular is seen by the following example:

$$P=\{p_n|p_{2k}=1+1/k, \quad p_{2k+1}=1+1/k\}.$$

Every other term of $\mathcal{C}(P)$ is infinite, yet P converges. Although $\mathcal{C}$ is not a regular transformation it does have the property of *joint convergence*. A transformation $\mathcal{T}$ is said to have the property of joint convergence if the convergence of P and $\mathcal{T}(P)$ imply that P and $\mathcal{T}(P)$ have the same limit. The proof of the following theorem will be given in section 5.1 (see theorem 10).

THEOREM 4: *If both* P *and* $\mathcal{C}(P)=Q$ *converge then*

$$\lim_{n\to\infty} p_n=\lim_{m\to\infty} q_m.$$

3.5. Properties of the Transformation $\mathcal{D}$

The following theorem gives algebraic properties of $\mathcal{D}$ similar to those of $\mathcal{C}$. The proof is by direct evaluation.

THEOREM 5: $\mathcal{D}(aP+b)=a\mathcal{D}(P)+b$.

Consider the real sequence

$$P=\left\{\sum_{i=1}^{k} a_i\lambda_1^n\cos(\theta_i+n\phi_i)\right\}$$

with $|\lambda_i|>|\lambda_{i+1}|$. Expression (6) may be directly evaluated as the ratio of the following two terms:

$$\sum_{|i-j|+|j-l|>0} a_i a_j a_l(\lambda_i\lambda_j\lambda_l/\lambda_1^2)^n \times\cos[\theta_i+\theta_j+\theta_l+n(\phi_i+\phi_j+\phi_l)]H_{ijl}$$

and

$$\sum_{i=1,j=1}^{k} a_i a_j(\lambda_i\lambda_j/\lambda_1^2)^n G_{ij},$$

where H_{ijl} and G_{ij} are functions of the λ's, θ's, and ϕ's that are independent of n. H_{ijl} and G_{ij} are trigonometric functions with coefficients depending on powers of the λ's.

An examination of the above expression and a simplification of notation leads to the following theorem.

THEOREM 6. *Let* $P=\{p_0+\sum_{i=1}^{k}a_i\lambda_i^n\cos(\theta_i+n\phi_i)\}$, $|\lambda_i|>|\lambda_{i+1}|$. *Then the* n*th term of* $\mathcal{D}(P)$ *is*

$$p_0+\frac{\lambda_2^n\sum_{i=0}^{m}b_i\mu_i^n\cos(\alpha_i+n\beta_i)}{1+o(1)} \qquad \begin{matrix}\mu_0=1\\ |\mu_i|<1, \quad i\geq 1\end{matrix}$$

except for possibly a finite number of terms.

One may also establish the following result:

THEOREM 7: *Let* $P=\{p_0+\sum_{i=1}^{k}a_i\lambda_i^n\}$, $|\lambda_i|>|\lambda_{i+1}|$. *Then the* n*th term of* $\mathcal{D}(P)$ *is*

$$\frac{p_0+(\lambda')^n\sum_{i=0}^{m}b_i\mu_i^n}{1+o(1)} \qquad \begin{matrix}\mu_0=1\\ |\mu_i|<1,\ i\geq 1\\ \lambda'=\max(\lambda_3,\lambda_2^2/\lambda_1)\end{matrix}$$

except for possibly a finite number of terms.

No attempt has been made to analyze the effect of $\mathcal{D}$ on alternating rational sequences; however the examples indicate that $\mathcal{D}$ is quite effective for increasing the rate of convergence.

It is a simple matter to construct sequences for which $\mathcal{D}$ is not regular. It is probable that $\mathcal{D}$ has the joint convergence property, but this has not been established.

4. Comparison With Aitken's δ^2-process and its Generalization

The transformations $\mathcal{C}$ and $\mathcal{D}$ are similar to a family of sequence transformations first studied by Aitken [1,2]. These transformations have been studied since by Lubkin [4], Samuelson [7], and Shanks [8]. The most general and complete treatment is by Shanks. A brief outline of these transformations will be given.

Let P be given and set

$$\Delta p_n = p_{n+1} - p_n.$$

From P new sequences $Q_k = \{q_{k,n} | n = k, k+1, \ldots\}$ are formed by

$$q_{k,n} = \frac{\begin{vmatrix} p_{n-k} & \cdots & p_{n-1} & p_n \\ \Delta p_{n-k} & \cdots & \Delta p_{n-1} & \Delta p_n \\ \Delta p_{n-k-1} & \cdots & \Delta p_n & \Delta p_{n+1} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \Delta p_{n-1} & \cdots & & \Delta p_{n+k+1} \end{vmatrix}}{\begin{vmatrix} 1 & \cdots & 1 & 1 \\ \Delta p_{n-k} & \cdots & \Delta p_{n-1} & \Delta p_n \\ \Delta p_{n-k-1} & \cdots & \Delta p_n & \Delta p_{n+1} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \Delta p_{n-1} & \cdots & & \Delta p_{n+k-1} \end{vmatrix}}.$$

This defines a set of sequence transformations $\mathcal{E}_k$, by

$$Q_k = \mathcal{E}_k(P).$$

Shanks studies in some detail the transformations $\mathcal{E}_1, \mathcal{E}_1^2, \mathcal{E}_1' = \{p_1, q_{12}, q_{13}^{(2)}, \ldots, q_{1,k+1}^{(k)}, \ldots\}$, and $\mathcal{E}_d = \{q_{k,k} | k = 0,1,2, \ldots\}$. $q_{1n}^{(k)}$ denotes the nth term of $\mathcal{E}_1^k(P)$. q_{1n} is given explicitly by

$$p_{1n} = \frac{p_{n+1}p_{n-1} - p_n^2}{p_{n+1} + p_{n-1} - 2p_n},$$

which defines Aitken's δ^2-process.

These transformations are derived heuristically as follows: Given a segment of P, $p_{n-k}, \ldots, p_{n+k}$, assume that this segment is exactly of the form $\sum_{i=1}^{k} a_i b_i^n + c$. Then define $q_{k,n}$ to be this c. The basic principle is the same as for $\mathcal{C}$ and $\mathcal{D}$, but the determination of the approximation is different.

If $S = \{s_n | s_n = \frac{1}{2}(p_n + p_{n+1})\}$ then it is seen that

$$\mathcal{C}(P) = \mathcal{E}_1(S).$$

Likewise, a straightforward calculation shows

$$\mathcal{D}(P) = \mathcal{E}_2(S).$$

The reason for this relationship is simple. $\mathcal{E}_k$ has been found by solving eq (8) for c:

$$\sum_{i=1}^{k} a_i b_i^m + c = p_m, \qquad m = n-k, \ldots, n+k. \quad (8)$$

The fact that best Tchebycheff approximations are characterized by the alternation $k+1$ times of the error function leads to the following equations:

$$\sum_{i=1}^{k} a_i b_i^m + c = p_m + (-1)^m e, \qquad m = n, \ldots, n+2k+2$$

where e is the error of the best approximation. If one averages the mth and $(m+1)$st equations, then

$$\sum_{i=1}^{k} \frac{a_i(1+b_i)}{2} b_i^m + c = \frac{1}{2}(p_m + p_{m+1}), \qquad m = n, \ldots, n+2k+1.$$

These equations are of the same form as (8) except P has been replaced by S.

Aitken considers the application of $\mathcal{E}_1$ to sequences of the form $P = \{p_0 + \sum_{i=1}^{k} \alpha_i \lambda_i^n\}$ and establishes:

THEOREM 8 (Aitken): *Let* $P = \{p_0 + \sum_{i=1}^{k} \alpha_i \lambda_i^n\}$, $|\lambda_i| > |\lambda_{i+1}|$; *then*

$$\mathcal{E}_1(P) = \left\{ p_0 + \frac{(\lambda_2)^n \sum_{i=0}^{m} d_i \mu_i^n}{1 + o(1)} \right\} \qquad \begin{matrix} \mu_0 = 1 \\ |\mu_i| < 1 \quad i \geq 1. \end{matrix}$$

Thus for sequences of this type $\mathcal{E}_1$ and $\mathcal{C}$ are equally effective in order of magnitude. The coefficients of $(\lambda_2)^n$ for $\mathcal{E}_1$ and $\mathcal{C}$ are, respectively

$$\frac{\alpha_1\alpha_2(\lambda_1 - \lambda_2)^2}{(1-\lambda_1)^2}, \qquad \frac{\alpha_1\alpha_2(1+\lambda_2)(\lambda_1-\lambda_2)^2}{(1-\lambda_1)^2}.$$

$\mathcal{E}_1$ appears to be slightly superior if $\lambda_2 > 0$. Examples indicate that either $\mathcal{E}_1$ or $\mathcal{C}$ may be better but in any case the difference in the rates of convergence is small.

For alternating rational sequences the two transformations are no longer similar in effect. Shanks has established:

THEOREM 9 (Shanks): *Let*

$$P = \left\{ p_0 + (-1)^n \sum_{i=0}^{k_1} a_i n^i \Big/ \sum_{i=0}^{k_2} b_i n^i \right\},$$

$a_{k_1} \neq 0$, $b_{k_2} \neq 0$ *and* $k = k_1 - k_2$. *Then the* n*th term of* $\mathcal{E}_1(P)$ *is*

$$p_0 + \frac{a_{k_1} k (-1)^n n^{k-2}}{4 b_{k_2}} + O(n^{k-3})$$

with possibly a finite number of exceptions.

Thus $\mathcal{E}_1$ accelerates the convergence by a factor of $1/n^2$ whereas $\mathcal{C}$ accelerates it by a factor of $1/n^3$.

The comparative effectiveness of $\mathcal{E}_2$ and $\mathcal{D}$ may be judged from the examples given later.

Theorem 10 shows that $\mathcal{E}_1$ has the property of joint convergence [4].

THEOREM 10 (Lubkin): *If both* P *and* $\mathcal{E}_1(P)=Q$ *converge, then*

$$\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n$$

If P converges then so does $S=\{\frac{1}{2}(p_n+p_{n+1})\}$ and

$$\lim_{n\to\infty} p_n = \lim_{n\to\infty} s_n$$

Thus theorem 4 follows directly from theorem 10.

5. Examples

Some examples are given which illustrate various characteristics of $\mathcal{C}$ and $\mathcal{D}$. In some cases comparisons are made with $\mathcal{E}_1$ and $\mathcal{E}_2$. For more general comparisons see [4] where $\mathcal{E}_1$ is compared with several classical linear sequence transformations for a large number of examples.

The first two sets of examples illustrate the general behavior of $\mathcal{C}$ and $\mathcal{D}$ for various exponential and alternating rational sequences. Some comparisons are made with $\mathcal{E}_1$ and $\mathcal{E}_2$ here. The next set of examples shows the effect of increasing the largest λ in exponential sequences. The following two sets show the effects of lower order terms and the loss of significant figures. Detailed comments are given with each set of examples.

All of these examples were computed using double precision arithmetic with about 20 significant decimal digits.

5.1. Effect of $\mathcal{C}$ and $\mathcal{D}$ on Exponential Sequences

The following five examples are considered:

$$P_1=\{2^n+(1.2)^n+(.9)^n+(.8)^n\},$$
$$P_2=\{(.98)_n+(.95)^n+(.5)^n+(.2)^n\},$$
$$P_3=\{(.98)^n+(.95)^n\},$$
$$P_4=\{(.9)^n+(.8)^n\cos(.1n)\},$$
$$P_5=\{(.9)^n\cos(1.5n)+(.5)^n\cos(.25n)\}.$$

Table 1 gives results of $\mathcal{C}$, $\mathcal{C}^2$, $\mathcal{C}^3$, $\mathcal{C}^4$, and $\mathcal{D}$ for 33 terms of P_1. P_1 diverges quite rapidly and $\mathcal{C}(P_1)$ diverges at a much slower rate. The factors in $\mathcal{C}(P_1)$ larger than .5 are 1.2, .9, .8, .72, and .54. $\mathcal{C}^2(P_1)$ converges as can be seen from the list of factors in $\mathcal{C}^2(P_1)$ larger than .5. They are .9, .8, .72, .68, .6, and .54. The four largest factors of $\mathcal{C}_3(P_2)$ are .8, .72, .71, and .68 are the largest factor of $\mathcal{C}^4(P_1)$ is .72.

TABLE 1.

P_1	$\mathcal{C}(P_1)$	$\mathcal{C}^2(P_1)$	$\mathcal{C}^3(P_1)$	$\mathcal{C}^4(P_1)$	$\mathcal{D}(P_1)$
4. 900000000000000	2. 9151478	2. 7270889	2. 7433842	2. 7029023	2. 6785648
6. 890000000000000	2. 7741418	2. 7426402	2. 8808482	3. 0727475	2. 7348679
10. 969000000000000	2. 7169114	2. 6757849	3. 2921161	69. 180737	2. 7088105
19. 139300000000000	2. 7430500	2. 5368629	4. 8080981	72. 818042	2. 6047836
35. 406490000000000	2. 8532800	2. 3466523	140. 43599	−. 54172894	2. 4392636
67. 779569000000000	3. 0509855	2. 1287500	−2. 4381772	−. 04037382	2. 2345869
132. 271192900000000	3. 3426610	1. 9033466	−. 76363656	. 04995813	2. 0125084
260. 898056330000000	3. 7381380	1. 6847333	−. 28755651	. 06710598	1. 7902083
517. 681418569000000	4. 2508558	1. 4814607	−. 09493555	. 06620607	1. 5791104
1030. 64778904490	4. 8982545	1. 2976421	−. 00731646	. 06136882	1. 3855556
2055. 82979364889	5. 7023160	1. 1344196	. 03331326	. 05701478	1. 2121566
4105. 26724946147	6. 6002694	. 99116645	. 05084351	. 05513287	1. 0591391
8203. 00848270213	7. 8954805	. 86033340	. 05650783	. 05808964	. 92537882
16397. 1119380351	9. 3585517	. 75798632	. 05602878	. 07372670	. 80910184
32783. 6480970788	11. 128667	. 66412067	. 05246752	. 17953286	. 70831457
65554. 7018754061	13. 265229	. 58282068	. 04750712	−. 11327597	. 62104494
131094. 375400883	15. 839833	. 51238289	. 04208180	−. 03107621	. 54546427
262170. 791442315	18. 938651	. 45125306	. 03670509	−. 01478933	. 47993879
524320. 097496628	22. 665309	. 39811524	. 03164886	−. 00839710	. 42304196
1048614. 47070579	27. 144335	. 35183192	. 02704444	−. 00521395	. 37354352
2097198. 12376227	32. 525314	. 31143263	. 02294119	−. 00341676	. 33037177
4194359. 31199968	38. 987872	. 27699212	. 01934122	−. 00231783	. 29269020
8388674. 34190501	46. 747669	. 24510968	. 01622020	−. 00160176	. 25998290
16777295. 5813360	56. 063584	. 21789028	. 01353993	−. 00114340	. 23010570
33554527. 4717843	67. 246349	. 19392818	. 01125615	−. 00081105	. 20587142
67108978. 5430931	80. 668906	. 17270283	. 00932342	−. 00056122	. 18756025
134217865. 431120	96. 778842	. 15411687	. 00769611	−. 00044434	. 14815635
268435620. 898931	116. 11330	. 13758611	. 00633423	−. 00029985	. 12460304
536871109. 862244				. 00001497	
1073742061. 41994					
2147483932. 89072					
4294967637. 85702					
8589935002. 21781					

It is seen that the first values of $\mathcal{C}^k(P_1)$ behave erratically as k increases. $\mathcal{C}^4(P_1)$ is a very erratic function of n for small n. This is the typical behavior and indicates that care must be exercised in using repeated applications of $\mathcal{C}$.

The largest factor in $\mathcal{D}(P_1)$ is .9 and hence $\mathcal{D}(P_1)$ converges.

Only selected values from the other examples are given. In table 2 it is seen that $\mathcal{C}$ is not very effective in increasing the rate of convergence of P_2.

TABLE 2

$P_2=\frac{1}{3}[(.98)^n+(.95)^n+(.5)^n+(.2)^n]$

n	P_2	$\mathcal{C}(P_2)$	$\mathcal{C}^2(P_2)$	$\mathcal{D}(P_2)$
1	2.6300	1.6876	3.0466	1.2528
10	1.4168	.32965	.20621	.21299
20	1.0261	.18278	.....	.02770

This is due to the closeness of the two largest factors. In the next table it is seen that even with the lower order term deleted the convergence is quite slow even for $\mathcal{C}^4(P_3)$.

TABLE 3

$P_3=\frac{1}{2}[(.98)^n+(.95)^n]$

n	P_3	$\mathcal{C}(P_3)$	$\mathcal{C}^2(P_3)$	$\mathcal{C}^3(P_3)$	$\mathcal{C}^4(P_3)$	$\mathcal{D}(P_3)$
1	1.93	.29914	−.07157	−.08733	−.07864	.0
25	.88085	.15891	−.04624	−4.50128	.36843	.0
40	.57421	.10160	−.02795	.02408	.00322	.0
60	.34362	.05175	−.01084	.00256	.00004	.0

The next table shows that $\mathcal{D}$ is fairly effective on P_4. It is to be noted however that convergence is no longer monotonic even in the large range considered. The denominator has a zero between the 67th and 68th terms of $\mathcal{D}(P_4)$. This same phenomenon is present in $\mathcal{D}^2(P_4)$.

TABLE 4

$P_4=\frac{1}{2}[(.9)^n+(.8)^n\cos(.1n)]$

n	P_4	$\mathcal{C}(P_4)$	$\mathcal{D}(P_4)$	$\mathcal{D}^2(P_4)$
1	1.6960	.02354	.08971	.05260
8	.54735	.07107	.11293	−.47672
10	.40669	.06085	−1.8972	−.00081
20	.11678	.01132	−.00282	−.00287
30	.04117	−.00036	−.00051	−.00048
35	.02465	−.00042	−.00151	.00005
50	.00516	−.00002	3×10^{-6}	2×10^{-7}
60	.00180	4×10^{-5}	5×10^{-7}	3×10^{-7}
67	.00086	3×10^{-5}	3×10^{-6}	6×10^{-7}

The final table shows that $\mathcal{D}$ is very effective for P_5, while $\mathcal{C}$ is not.

TABLE 5

$P_5=\frac{1}{2}[(.9)^n\cos(1.5n)+(.5)^n\cos(.25n)]$

n	P_5	$\mathcal{C}(P_5)$	$\mathcal{D}(P_5)$	$\mathcal{D}^2(P_5)$
1	.54812	−.11751	.10538	−.00135
5	.21454	−.21059	.00017	-1×10^{-6}
10	−.26567	3.2919	−.00022	-6×10^{-8}
15	−.17983	.07227	-5×10^{-6}	5×10^{-11}
20	.01875	.06441	1×10^{-7}	6×10^{-10}
25	.07037	−.04624	7×10^{-9}	.....

5.2. Effect of $\mathcal{C}$ and $\mathcal{D}$ on Alternating Sequences

Consider the sequences of partial sums of the following series:

$$\pi=4\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{2i-1}=3.1415926,$$

$$ln\,2=\sum_{i=1}^{\infty}\frac{(-1)^i}{i}=.6931472,$$

$$ln\,3=\sum_{i=1}^{\infty}\frac{(-1)^i 2^i}{i}=1.0986\,12.$$

These sequences are denoted by A_1, A_2, and A_3 respectively.

The first two examples are alternating rational sequences and tables 6 and 7 show the results for $\mathcal{C}$ and $\mathcal{E}_1$ to be as predicted by theorems 3 and 9. $\mathcal{D}(A_1)$, $\mathcal{D}(A_2)$, $\mathcal{E}_2(A_1)$, and $\mathcal{E}_2(A_2)$ converge much more rapidly and apparently $\mathcal{D}$ accelerates the convergence better than $\mathcal{E}_2$ by a factor in $1/n$.

TABLE 6

n	A_1	$\mathcal{C}(A_1)$	$\mathcal{E}_1(A_1)$	$\mathcal{D}(A_1)$	$\mathcal{E}_2(A_1)$
1	4.0	3.1466666	3.1666667	3.1417367	3.1423423
2	2.6666667	3.140136	3.1333333	3.1415567	3.1413919
3	3.4666667	3.1421516	3.1452381	3.1416042	3.1416627
4	2.8952381	3.1413354	3.1896825	3.1415882	3.1415634
5	3.3396825	3.1417266	3.1427129	3.1415946	3.1416065
6	2.9760462	3.1415163	3.1408813	.....	3.1415854
7	3.2837385	3.1416392	3.1420718	.....	.....
8	3.0170718	.....	3.1412548	.....	.....
9	3.2523659	.....	.....	.....	.....
10	3.0418396	.....	.....	.....	.....
error of last term.....	.099753	.0000466	.0003378	.0000020	.0000072

TABLE 7

n	A_2	$\mathcal{C}(A_2)$	$\mathcal{E}_1(A_2)$	$\mathcal{D}(A_2)$	$\mathcal{E}_2(A_2)$
1	1.0	.69444444	.7	.69318182	.69333333
2	.5	.69270833	.69047619	.69313724	.69308943
3	.83333333	.69333333	.69444444	.69315068	.69316940
4	.58333333	.69305555	.69242424	.69314573	.69313724
5	.78333333	.69319726	.69358974	.69314783	.69315212
6	.61666666	.69311755	.69285712	.69314682	.69314448
7	.75952379	.69316576	.69334732	.69314734	.69314872
8	.63452379	.69313490	.69300331	.69314706	.69314621
9	.74563490	.69315556	.69325395	.69314723	.69314778
10	.64563490	.69314123	.69306573	.69314713	.69314677
11	.73654400	.69315149	.69321067	.69314719	.69314745
12	.65321067	.69314395	.69309670	.69314715	.69314698
13	.73013375	.69314961	.69318793	.69314719	.69314731
14	.65870517	.69314529	.69311377	.....	.69314707
15	.72537184	.69314865	.69317488	.....	.....
16	.66287184	.....	.69312394	.....	.....
17	.72169538	.....	.....	.....	.....
18	.66613982	.....	.....	.....	.....
error of last term.....	.0220074	.0000015	.0000232	.00000001	.00000010

It is remarkable that π can be found so accurately from a few terms of the simple sequence A_1. Shanks considers $\mathcal{E}_1^k(A_1)$ and π is given to 8 places by the first term of $\mathcal{E}_1^4(A_1)$. It would require approximately 40,000,000 terms of A_1 to give the same accuracy.

The third example is a "mixed" sequence. All of the transformed sequences diverge, but they are *semiconvergent* or *asymptotic* [3, pp. 520, 536] sequences for $\ln 3$. In this example $\mathcal{C}$ and $\mathcal{D}$ show a very marked improvement over $\mathcal{E}_1$ and $\mathcal{E}_2$. It appears that round off has begun to affect $\mathcal{D}(A_2)$. Although $\mathcal{D}(A_2)$ was computed using double precision, A_2 itself was computed to only eight significant figures.

TABLE 8

n	A_n	$\mathcal{C}(A_n)$	$\mathcal{E}_1(A_n)$	$\mathcal{D}(A_n)$	$\mathcal{E}_2(A_n)$
1	0	1.0	1.0	1.0980392	1.0909091
2	*2.0	1.1111111	1.1428571	1.0980392	1.1014493
3	0	1.0952381	1.0666666	1.1047618	1.0970464
4	2.6666667	*1.0986667	*1.1282052	1.0991870	1.0997246
5	−1.3333333	1.1008547	1.0666666	1.0984127	1.0976744
6	5.0666667	1.0938776	1.1308123	1.0986835	1.0995076
7	−5.6	1.1067668	1.0933503	*1.0985975	*1.0976739
8	12.685714	1.0852653	1.1057143	.....	1.0996695
9	−19.314286	1.1202030	1.0931748	.....	.....
10	37.574603	.....	1.2391205	.....	.....
11	−64.825397	.....	.....	.....	.....
12	121.35642	.....	.....	.....	.....
error of last term	120.254	.0215908	.140508	.0000147	.0010573
*min error	.831388	.0000545	.029593	.0000147	.0008454

The sequence A_1 may be modified by repeating each term in the sequence twice. Call the resulting sequence A_1'. The application of $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}_1$ and $\mathcal{E}_2$ to A_1' give some unusual results. It is seen that

$$\mathcal{E}_1(A_1') = A_1'$$

$$\mathcal{C}(A_1') = \{a_n | a_n = \infty, \quad n = 1, 2, \ldots\}.$$

$\mathcal{E}_2(A_1')$ is of the same type as A_1' with each term repeated twice. The odd terms of $\mathcal{D}(A_1')$ agree with those of $\mathcal{E}_2(A_1')$, but the even terms do not. The even terms of $\mathcal{E}(A_1')$ converge much more rapidly than the odd ones, by a factor of approximately $1/n$. Both $\mathcal{E}_2(A_1')$ and $\mathcal{D}(A_1')$ converge more slowly than $\mathcal{E}_2(A_1)$ and $\mathcal{D}(A_1)$.

5.3. Three Related Sequences

The following three sequences are considered:

$$P_3 = \{(.98)^n + (.95)^n\},$$
$$P_6 = \{(1.96)^n + (.95)^n\},$$
$$P_7 = \{(1.96)^n + (1.9)^n\}.$$

The purpose is to investigate the following two questions. Is it advantageous to increase the largest term in an exponential sequence before applying $\mathcal{C}$? Is it advantageous to keep the larger terms away from 1 at the expense of increasing them? The answer to the first question appears to be sometimes and the answer to the second appears to be no.

The limit or antilimit of each of these sequences is zero. The first converges quite slowly and the other two diverge (see table 9).

TABLE 9. *Behavior of P_3, P_6, and P_7*

n	P_3	P_6	P_7
10	1.416	837.3	1450
25	.8808	2.025×10^{7}	2.955×10^{7}
50	.4411	4.1×10^{13}	4.967×10^{13}
75	.2411	.....	.....

P_3, $\mathcal{C}(P_3)$, $\mathcal{C}^2(P_3)$, and $\mathcal{C}^3(P_3)$ all converge slowly. There is an improvement by a factor of approximately 2 in the range considered for each further application of $\mathcal{C}$.

Theorem 2 states that $\mathcal{C}(P_3)$ and $\mathcal{C}(P_6)$ converge at the same rate while $\mathcal{C}(P_7)$ diverges. This is borne out by table 10. $\mathcal{C}^2$ and $\mathcal{C}^3$ produce entirely different effects for each sequence. $\mathcal{C}^2(P_6)$ converges very rapidly to zero and $\mathcal{C}^3(P_6)$ converges even more rapidly. Errors due to the loss of significant figures have affected the 24th term of $\mathcal{C}^2(P_6)$ and the 14th term of $\mathcal{C}^3(P_6)$. All calculations here were made with 20 decimal digits.

In P_7, $\mathcal{C}(P_7)$, $\mathcal{C}^2(P_7)$, and $\mathcal{C}^3(P_7)$ the first term is the closest to zero. However it is apparent that the sequence formed by the nth terms of $\mathcal{C}^k(P_7)$, $k = 0, 1, \ldots$ converges quite rapidly to the antilimit of P_7.

5.4. Effect of Lower-Order Terms

To illustrate the effect of lower order terms we consider

$$P_8 = \{2^n + (1.2)^n\},$$

which may be compared to P_1 in section 6.2.

TABLE 10. *Effect of $\mathbf{C}^k$ on the sequences $P_3 = \{(.98)^n + (.95)^n\}$, $P_6 = \{(1.96)^n + (.95)^n\}$, $P_7 = \{(1.96)^n + (1.9)^n\}$.*

n	$\mathcal{C}(P_3)$	$\mathcal{C}^2(P_3)$	$\mathcal{C}^3(P_3)$	$\mathcal{C}(P_6)$	$\mathcal{C}^2(P_6)$	$\mathcal{C}^3(P_6)$	$\mathcal{C}(P_7)$	$\mathcal{C}^2(P_7)$	$\mathcal{C}^3(P_7)$
1	.2991	−.0716	−.0873	1.024	−.0671	$-.55\times10^{-3}$	.0059	$-.88\times10^{-5}$	$.40\times10^{-7}$
5	.2708	−.0684	−.0523	.8359	−.0028	$-.13\times10^{-3}$	.0807	$-.12\times10^{-3}$	$.54\times10^{-6}$
10	.2384	−.0638	−.1032	.6462	$-.57\times10^{-4}$	$-.75\times10^{-9}$	2.129	−.0031	$.13\times10^{-4}$
15	.2091	−.0584	−.1269	.5000	$-.12\times10^{-5}$	$-.12\times10^{-10}$	55.83	−.0769	$.32\times10^{-3}$
20	.1827	−.0525	−.2017	.3869	$-.25\times10^{-7}$	$-.21\times10^{-9}$	1456	−1.901	.0076
25	.1589	−.0462	−4.903	.2994	$.15\times10^{-8}$	$-.19\times10^{-8}$	$.38\times10^{5}$	−46.27	.1726
30	.1376	−.0399	.1319	.2316	$.15\times10^{-7}$	$-.38\times10^{-8}$	$.98\times10^{6}$	−1109	3.852
35	.1186	−.0337	.0450	.1792	$.65\times10^{-7}$	$-.98\times10^{-6}$	$.25\times10^{8}$	$-.26\times10^{5}$	84.02
40	.1016	−.0279	.0241	.1387	$.94\times10^{-3}$	$-.12\times10^{-4}$	$.64\times10^{9}$	$-.61\times10^{6}$	1795
45	.0866	−.0227	.0131	.1073	$.75\times10^{-4}$	.....	$.16\times10^{11}$	$-.14\times10^{8}$	.....

TABLE 11. *Effect of lower order terms*

n	P_1	P_8	$\mathcal{C}(P_1)$	$\mathcal{C}(P_8)$	$\mathcal{C}^2(P_1)$	$\mathcal{C}^2(P_8)$	$\mathcal{C}^3(P_1)$	$\mathcal{C}^3(P_8)$
1	4.8426	3.1625	2.8810	.8205	2.635	−.0719	2.711	2×10^{-4}
10	1018.569885	1018.119177	4.8408	4.3071	1.282	−.00378	−.0072	3×10^{-7}
20	10,363,260.20	10,363,258.88	26.826	26.673	.3477	−.00014	.0267	5×10^{-9}
27	1,326,449,998	1,326,449,997	95.645	95.576				

In table 11 it is seen that the lower order terms disappear rather rapidly in P_1, but the 27th terms of $\mathcal{C}(P_1)$ and $\mathcal{C}(P_8)$ only agree to two digits. Further $\mathcal{C}^2(P_1)$ and $\mathcal{C}^3(P_1)$ are completely unrelated to $\mathcal{C}^2(P_8)$ and $\mathcal{C}^3(P_8)$. Thus it is seen that these lower order terms are significant even for $\mathcal{C}(P)$ and they affect the rate of convergence of $\mathcal{C}^k(P)$, $k>1$, drastically and adversely.

5.5. Loss of Significant Figures

It has been seen that terms with a small effect on P can have a large effect on $\mathcal{C}^k(P)$. A related phenomenon is the loss of significant figures in the computation of $\mathcal{C}(P)$ and $\mathcal{D}(P)$. If P is the constant sequence $\{p_n|p_n=1\}$, then $\mathcal{C}(P)$ and $\mathcal{D}(P)$ may be written symbolically as

$$\frac{1\cdot(1-1)}{(1-1)}$$

Hence slight inaccuracies in the computation of $\mathcal{C}(P)$ and $\mathcal{D}(P)$ may distort the results badly if P is slowly varying. Often it is necessary to use double precision for the computation of $\mathcal{D}(P)$ even though P is only given to six or eight digits.

The following table shows where the loss of significant figures can occur. The first column of table 12 defines the sequence P, the second column gives the approximate size of the terms of P in the range that the transformation is applied and the third column gives the transformation used. The next four columns give n, the size of the terms appearing in the numerator, the actual value of the numerator and the number of significant figures lost from P.

TABLE 12. *Loss of significant figures*

Sequence	Size of terms	Transformation	n	Size of numerical terms	Numerator	Significant figures lost
$(1.96)^n+(.95)^n$	$.1\times10^9$	$\mathcal{C}^2$	25	.01	$.43\times10^{-4}$	20
$(.9)^n+(.8)^n\cos(.1n)$	$.4\times10^{-1}$	$\mathcal{D}$	30	.0002	$.3\times10^{-10}$	9
$(.9)^n+(.8)^n\cos(.1n)$	$.5\times10^{-3}$	$\mathcal{D}$	75	$.2\times10^{-3}$	$.2\times10^{-21}$	18
$(.999)^n+(.99)^n$	2	$\mathcal{D}$	all	10	10^{-18}	19
$(.98)^n+(.6)^n+(.4)^n$	1	$\mathcal{D}$	15	1	10^{-11}	11

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Numerical Solution of the Frequency Equations for the Flexural Vibration of Cylindrical Rods

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A numerical solution of Pickett's frequency equations for the flexural vibration of cylindrical rods, based on the three-dimensional differential equations of elasticity, has been obtained on the IBM 704 computer. The results are presented in the form of tables of correction factors to be applied to the thin rod approximation for the fundamental flexural resonance frequency and the first two overtones. These results provide an accurate means of calculating Young's modulus from the density, dimensions, and resonance frequencies of cylindrical rods having diameter-to-length ratios as high as 0.6.

1. Introduction

Improved experimental techniques recently have made it possible to determine the elastic properties of materials to a high degree of precision by dynamic resonance methods. Factors which have limited accurate measurements are: Convenient experimental techniques are not well developed theoretically, and conversely, techniques based on well analyzed theories are experimentally difficult. A recent experimental investigation [1] [1] showed that Pickett's [2] approximate solution of the flexure problem, based on Goen's [3] solution of Timoshenko's [4] equations may be in error by a few percent. In a later paper, Pickett [5] gave an accurate set of frequency equations based on the three-dimensional differential equations of elasticity. The purpose of the present paper is to present the results of a numerical solution of these more accurate equations. These numerical solutions have put the accurate theory of flexural vibration of cylinders into a more usable form. Elimination of theoretical errors, for this particular problem, has thus been achieved.

2. Background

According to the elementary, one-dimensional theory, Young's modulus, E, may be computed from the flexural resonance frequency, f, of a cylinder of density ρ, length l, and diameter d, from the equation [3]

$$E=\frac{64\pi^2\rho l^4 f_n^2}{m_n^4 d^2}, \tag{1}$$

where f_n is the resonance frequency of the n^{th} mode of vibration and the m_n are solutions of the equation

$$\cos m \cosh m = 1.$$

The values of m_n for the first three modes of vibration are [6]

$$m_1 = 4.730041, \tag{3}$$

$$m_2 = 7.853205, \tag{4}$$

and

$$m_3 = 10.995608. \tag{5}$$

Timoshenko [4] added shear and rotatory inertia corrections to the differential equation, and Goens [3] reduced the resulting frequency equation to the form

$$E=\frac{64\pi^2\rho l^4 f_n^2}{m_n^4 d^2}\, T_n, \tag{6}$$

where T_n is a function of d/l and Poisson's ratio, μ. T_n approaches one as d/l approaches zero, for all values of n and μ.

Pickett [2] made numerical computations of the T_n, and gave approximate equations for this computation. These values are not as accurate as is desirable, in view of the experimental precision now attainable in dynamic resonance measurements. The basis for the more accurate values presented in this paper was given by Pickett [5] in his solution based upon the three-dimensional differential equations of elasticity.

A brief verbal sketch of Pickett's derivation may be helpful in understanding the meaning of some of the following equations. He begins by noting that there are three fundamental solutions for elastic waves in a cylinder. Each is characterized by the angular frequency p and the wavelength $2\pi a/\gamma$, where a is the radius of the cylinder. These three solutions can be combined so that their sum satisfies the boundary conditions on the curved surface of the cylinder. This gives a relationship between p and γ expressed by Pickett's [5] equation 12. For each value of p there are two values of γ, denoted by γ and γ', which satisfy this equation. One of these, γ', corresponds to a sum which represents a traveling wave. The other, γ, corresponds to a sum in which the "wavelength" is an imaginary number, i.e., the displacement is an exponential, rather than a sinusoidal function of distance along the rod. Pickett combines both of these sums to get a solution for displacements which approximately satisfies the boundary conditions on the flat ends of the cylinder. These boundary conditions lead to his equation 21, the frequency condition. The present paper is concerned with the simultaneous numerical solution of these equations.

[1] Figures in brackets indicate the literature references at the end of this paper

3. Theory and Calculations

Pickett's [5] results were reduced to the form of equations (7) through (12) below, by a method similar to that used by Hudson [7]. Pickett's equation 12 is an expression of the requirement that the three radial components of stress be zero at the surface of the cylinder. This equation was reduced to the form of eq (7) by the aid of definition (10) and a suitable manipulation of the resulting determinant to reduce it from third order to second order.

$$\begin{vmatrix} 2K^2\psi(h)+(K^2-\gamma^2)[2\psi(K)+K^2] & -2\gamma^2[3\psi(K)+K^2] \\ 2K^2\psi(h)+(K^2+\gamma^2) & (K^2-\gamma^2)\psi(K)+(K^2+\gamma^2 \end{vmatrix} = 0, \quad (7)$$

where h and K are defined by

$$h=\left[\frac{(1+\mu)(1-2\mu)}{1-\mu}\frac{\rho a^2p^2}{E}-\gamma^2\right]^{\frac{1}{2}}, \quad (8)$$

$$K=\left[2(1+\mu)\frac{\rho a^2p^2}{E}-\gamma^2\right]^{\frac{1}{2}}, \quad (9)$$

where

μ = Poisson's ratio,
E = Young's modulus,
ρ = density,
$a=d/2$, d being diameter of cylinder,
$p=2\pi f$ = angular frequency of flexural vibration,
γ = parameter determined from the solution of these equations,
ψ = a combination of Bessel functions of the first kind, orders zero and one, and is defined by

$$\psi(X)=\frac{XJ_0(X)}{J_1(X)}-2$$

$$=-\frac{X^2}{4}\left[1+\frac{X^2}{24}+\frac{X^4}{384}+\frac{X^6}{5760}+\cdots\right]. \quad (10)$$

If we define the function

$$F(\gamma)=\frac{K^2\gamma}{\gamma^2+K^2}\left[\psi(h)K^2\left(\frac{K^2+\gamma^2}{2h^2}-1\right)+\psi(K)\left(\frac{K^2-\gamma^2}{2}\right)+\psi(h)\psi(K)\left(3\frac{K^2+\gamma^2}{2h^2}-2\right)+\psi^2(K)\left(\frac{K^2-\gamma^2}{K^2}\right)\right]\left[K^2\psi(h)+(K^2+2\gamma^2)\psi(K)+\frac{K^2}{2}(\gamma^2+K^2)\right]^{-1}\begin{Bmatrix}-\cot\frac{\gamma l}{2a}\\ \tan\frac{\gamma l}{2a}\end{Bmatrix}, \quad (11)$$

where the upper terms in braces refer to the odd numbered and the lower terms to the even numbered modes of vibration, then equation 21 of [5] becomes

$$F(\gamma)=F(\gamma'). \quad (12)$$

The problem of determining the flexural resonance frequencies is now solved in principle. For a given length, diameter, Young's modulus, and Poisson's ratio, one must solve (7) and (12) simultaneously for p (and incidentally for γ and γ'). However, these are transcendental equations and involve the frequency in such a complex way that numerical methods must be used.

A program for obtaining a numerical solution of the simultaneous set of transcendental eqs (7) and (12), using (11), on the IBM 704 computer was written by the Bureau's computer section. The numerical solution was made in the manner suggested by Pickett [5]. Appropriate values of $\rho a^2p^2/E$ and μ were chosen. The two real values of γ^2 satisfying eq (7) were determined by successive approximations. These values of γ^2 were then substituted in eq (11) and the values of $l/2a$ satisfying eq (12) for each of the first three modes of vibration were determined by successive approximation. The correction factors, T_n, were then computed from the relation

$$T_n=\frac{m_n^4\left(\frac{a}{l}\right)^4}{4\frac{\rho a^2p^2}{E}}. \quad (13)$$

Twenty-nine values of $\rho a^2p^2/E$ were chosen, ranging in value from 1.0×10^{-6} to 0.3958. For each of these values of $\rho a^2p^2/E$, eleven values of μ were chosen, ranging from zero to 0.50 in steps of 0.05. There were thus 319 computed points for each of the first three modes of vibration.

The values of T_1, T_2, and T_3 were then computed for equal intervals of $2a/l$ by a sixth-order Aitken's [8] interpolation method. The overall error in these computations was less than three parts in 10^6 for values of $\rho a^2p^2/E$ greater than about 0.0007, corresponding to $d/l\approx0.1$ for the fundamental mode. For lower values of $\rho a^2p^2/E$, however, machine round-off error became significant. In order to obtain accurate values of T for $\rho a^2p^2/E<0.0007$, which corresponds to long thin rods, it was necessary to make a first-order approximation to the set of eqs (7), (11), and (12). This was accomplished as follows:

If we let

$$\frac{(1+\mu)(1-2\mu)}{1-\mu}=A, \quad 2(1+\mu)=B, \text{ and } \frac{\rho a^2p^2}{E}=P,$$

then, from eqs. (8) and (9),

$$h^2=AP-\gamma^2, \qquad K^2=BP-\gamma^2, \quad (14)$$

and from eq (10) we obtain

$$\psi(h)=-\frac{AP-\gamma^2}{4}\left[1+\frac{AP-\gamma^2}{24}+\frac{(AP-\gamma^2)^2}{384}+\frac{(AP-\gamma^2)^3}{5760}+\cdots\right] \quad (15)$$

and

$$\psi(K)=-\frac{BP-\gamma^2}{4}\left[1+\frac{BP-\gamma^2}{24}+\frac{(BP-\gamma^2)^2}{384}+\frac{(BP-\gamma^2)^3}{5760}+\cdots\right] \quad (16)$$

If eqs (14), (15), and (16) are substituted in eq (7), terms of order γ^2 cancel (terms of the form $\gamma^2\rho^n$ are higher order). Inspection of the lowest order terms present shows that

$$\gamma^2\approx\pm 2\sqrt{P}. \quad (17)$$

But obtaining a solution to (12) requires that the magnitude of γ^2 and γ'^2 be different. We must, therefore, make a second-order approximation to eq (7). Also, it is obvious from eq (17) that γ^2 is of order $\sqrt{P}$, not of order P; therefore, we must keep the first three terms of (15) and (16) for this approximation. When this is done, and algebraic simplifications are made, we arrive at

$$\gamma^4-4P-\frac{2}{3}(5+2\mu)P\gamma^2-\frac{\mu}{12(1+\mu)}\gamma^6=0, \quad (18)$$

where the last two terms are of higher order than the first two. In order to solve (18) for the two real values of γ^2, assume

$$\gamma^2=C\sqrt{P}+DP, \quad (19)$$

where C and D are functions of μ to be determined. The substitution of (19) into (18) results in a polynomial in $\sqrt{P}$. If we neglect terms of order higher than $p^{3/2}$, and equate the coefficients of each of the remaining powers of P separately to zero, we arrive at the following values for C and D.

$$C=\pm 2$$

and

$$D=\frac{10+15\mu+4\mu^2}{6(1+\mu)}.$$

Therefore,

$$\gamma'^2=+2\sqrt{\frac{\rho a^2p^2}{E}+\left(\frac{10+15\mu+4\mu^2}{6(1+\mu)}\right)\left(\frac{\rho a^2p^2}{E}\right)}, \quad (20)$$

and

$$\gamma^2=-2\sqrt{\frac{\rho a^2p^2}{E}+\left(\frac{10+15\mu+4\mu^2}{6(1+\mu)}\right)\left(\frac{\rho a^2p^2}{E}\right)}. \quad (21)$$

Equations (20) and (21) were then substituted in (11) and (12), and the functions $\cot \gamma l/2a$, $\cot \gamma' l/2a$, $\tan \gamma l/2a$, and $\tan \gamma' l/2a$ were approximated in the following manner: since $\gamma^2<0$, γ is imaginary, so the following relations hold identically:

$$\cot\frac{\gamma l}{2a}=-i\coth\frac{|\gamma|l}{2a}$$

and

$$\tan\frac{\gamma l}{2a}=i\tanh\frac{|\gamma|l}{2a}.$$

Furthermore, it is known from the elementary solution that

$$\lim_{\frac{2a}{l}\to 0}\cot\frac{\gamma' l}{2a}=\lim_{\frac{2a}{l}\to 0}\left(-\coth\frac{|\gamma|l}{2a}\right)=\cot\frac{m}{2}$$

for odd-numbered modes, and

$$\lim_{\frac{2a}{l}\to 0}\tan\frac{\gamma' l}{2a}=\lim_{\frac{2a}{l}\to 0}\tanh\frac{|\gamma|l}{2a}=\tan\frac{m}{2}$$

for even-numbered modes. We therefore make a first-order Taylor's series expansion of these functions about the point $m/2$, giving

$$\cot\frac{\gamma' l}{2a}\approx\cot\frac{m}{2}-\frac{\frac{\gamma' l}{2a}-\frac{m}{2}}{\sin^2\frac{m}{2}},$$

and similar approximations for the other three functions.

The quantity $\rho a^2p^2/E$ was eliminated by means of (13), and the various powers of T which appeared were eliminated by means of a Taylor's series expansion about the point $d/l=0$, $T=1$. This expansion can be shown to contain no terms of order d/l, so the first correction term in the expansion is proportional to $(d/l)^2$.

With these substitutions and further algebraic and trigonometric manipulation, we arrive at the following equation

$$T\approx 1+\left[\left(\frac{m^2(\sin m+\sinh m)}{\sinh m-\sin m}\right)\left(\frac{10+15\mu+4\mu^2}{48(1+\mu)}\right)\pm\left(\frac{m\sin m\sinh m}{\sin m-\sinh m}\right)\left(\frac{2-3\mu-4\mu^2}{24(1+\mu)}\right)\right]\left(\frac{d}{l}\right)^2,$$

where the upper (plus) sign refers to odd-numbered modes and the lower (minus) sign refers to even-numbered modes of vibration.

Inserting the values of m from eq (3), (4), and (5) we obtain

$$T_1\approx 1+4.88669\left(\frac{1+1.26225\mu+0.20980\mu^2}{1+\mu}\right)\left(\frac{d}{l}\right)^2, \quad (22)$$

$$T_2\approx 1+13.5234\left(\frac{1+1.35472\mu+0.28377\mu^2}{1+\mu}\right)\left(\frac{d}{l}\right)^2, \quad (23)$$

$$T_3\approx 1+26.1028\left(\frac{1+1.39469\mu+0.31575\mu^2}{1+\mu}\right)\left(\frac{d}{l}\right)^2. \quad (24)$$

A plot of T versus $(d/l)^2$ is nearly a straight line. This indicates that the first term (of order $(d/l)^2$) in the Taylor's series expansion of T gives most of the contribution over the range of interest. Thus, the function

$$H=\frac{T-1}{\left(\frac{d}{l}\right)^2} \quad (25)$$

varies only very slowly with $(d/l)^2$

The method for obtaining accurate values of T in the region $(d/l)^2 << 1$ is now apparent. A plot of H versus $(d/l)^2$ is made on a highly expanded scale, for each of the first three modes of vibration and for each value of μ, with the H intercept being obtained in each case from the appropriate one of eq (22), (23), or (24) and the other points being taken from the computer solution. When plotted in this manner, the first few points from the computer solution did not form a smooth curve. This scattering of points was an indication of significant round-off error at these low values of $(d/l)^2$. These points were therefore ignored, and the smooth curve, obtained at higher values of $(d/l)^2$, was extended to the chosen intercept. Finally, the values of H for equally spaced intervals of (d/l) were read from this smoothed curve, and the values of T were computed from eq (25).

4. Results and Discussion

Tables 1, 2, and 3 give T as a function of d/l and μ, for the first, second, and third modes respectively. These tables are accurate to about three parts in 10^6. In order to calculate Young's modulus from measurements of the flexural resonance frequencies of cylinders, the following equations, based on eq (6), are used:

$$E=1.261886\frac{\rho l^4 f_1^2}{d^2}T_1,$$

$$E=0.1660703\frac{\rho l^4 f_2^2}{d^2}T_2,$$

and

$$E=0.04321184\frac{\rho l^4 f_3^2}{d^2}T_3,$$

for the first, second, and third modes respectively.

A linear interpolation of T versus $(d/l)^2$ or T versus μ can be made with an accuracy of better than one part in 10^4 over the range of the tables. To obtain higher accuracy, Aitken's interpolation method should be used.

In order to obtain maximum accuracy from this method of Young's modulus determination, it is recommended that, whenever feasible, resonance frequencies of the fundamental and first two overtones be measured, and the values of E be computed for each. An average of these values of E should be more accurate than any of them alone. The reason for this is that small irregularities in the diameter of a specimen, which can never be eliminated completely, will affect each mode differently; i.e., some modes will have their frequencies shifted upward, and some downward. The magnitude of the disagreement among these values of E indicates the overall uniformity of the dimensions of the specimen.

The correctness of these computations was checked experimentally, by a method which will be described in a subsequent publication. Table 4 gives a comparison of experimental and theoretical values of T for two of the most accurately machined specimens. The discrepancy between theory and experiment is seen to be of the order of 0.03 percent, except for T_3. The experimental value of T_3 was difficult to

TABLE 1. *The correction factor for the fundamental mode of flexural vibration of cylinders, T_1, as a function of the diameter to length ratio, d/l, and Poisson's ratio, μ.*

The values of T_1 for $d/l \leq 0.08$ were obtained by a graphical interpolation as explained in the text. All other values of T_1 were obtained from the computer solution.

d/l \ μ	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
0.00	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
.02	1.001954	1.001979	1.002004	1.002029	1.002053	1.002077	1.002100	1.002124	1.002147	1.002170	1.002193
.04	1.007804	1.007905	1.008004	1.008102	1.008199	1.008295	1.008388	1.008482	1.008575	1.008666	1.008757
.06	1.017522	1.017748	1.017968	1.018186	1.018405	1.018619	1.018826	1.019038	1.019245	1.019450	1.019653
.08	1.031064	1.031461	1.031848	1.032233	1.032618	1.032994	1.033360	1.033733	1.034096	1.034459	1.034818
.10	1.048367	1.048983	1.049580	1.050174	1.050765	1.051344	1.051916	1.052484	1.053050	1.053610	1.054170
.12	1.069357	1.070225	1.071077	1.071920	1.072753	1.073577	1.074393	1.075202	1.076008	1.076808	1.077606
.14	1.093950	1.095111	1.096256	1.097378	1.098495	1.099599	1.100694	1.101782	1.102864	1.103941	1.105015
.16	1.122061	1.123544	1.125007	1.126452	1.127884	1.129302	1.130711	1.132113	1.133509	1.134901	1.136288
.18	1.153592	1.155430	1.157245	1.159039	1.160817	1.162582	1.164337	1.166086	1.167830	1.169569	1.171308
.20	1.188458	1.190676	1.192869	1.195038	1.197191	1.199332	1.201464	1.203591	1.205714	1.207838	1.209961
.22	1.226567	1.229189	1.231781	1.234351	1.236906	1.239449	1.241986	1.244521	1.247057	1.249596	1.252140
.24	1.267838	1.270881	1.273894	1.276886	1.279865	1.282836	1.285807	1.288779	1.291757	1.294746	1.297747
.26	1.312189	1.315668	1.319120	1.322555	1.325980	1.329403	1.332832	1.336270	1.339723	1.343194	1.346685
.28	1.359546	1.363477	1.367381	1.371276	1.375167	1.379065	1.382977	1.386911	1.390869	1.394857	1.398879
.30	1.409842	1.414233	1.418606	1.422974	1.427352	1.431747	1.436169	1.440625	1.445121	1.449661	1.454250
.32	1.463014	1.467873	1.472724	1.477584	1.482465	1.487380	1.492337	1.497345	1.502411	1.507540	1.512735
.34	1.519006	1.524340	1.529680	1.535043	1.540446	1.545901	1.551420	1.557012	1.562682	1.568439	1.574286
.36	1.577766	1.583580	1.589417	1.595298	1.601240	1.607259	1.613367	1.619572	1.625884	1.632309	1.638853
.38	1.639253	1.645548	1.651888	1.658300	1.664800	1.671405	1.678129	1.684983	1.691975	1.699113	1.706401
.40	1.703422	1.710202	1.717054	1.724007	1.731082	1.738298	1.745669	1.753207	1.760921	1.768818	1.776904
.42	1.770243	1.777505	1.784874	1.792382	1.800052	1.807904	1.815954	1.824214	1.832694	1.841401	1.850342
.44	1.839681	1.847424	1.855317	1.863393	1.871677	1.880193	1.888956	1.897979	1.907273	1.916846	1.926702
.46	1.911712	1.919934	1.928355	1.937012	1.945932	1.955140	1.964653	1.974485	1.984646	1.995143	2.005986
.48	1.986311	1.995009	2.003964	2.013216	2.022795	2.032727	2.043030	2.053718	2.064802	2.076288	2.088178
.50	2.063458	2.072628	2.082123	2.091985	2.102247	2.112937	2.124075	2.135673	2.147741	2.160285	2.173306
.52	2.143136	2.152775	2.162814	2.173303	2.184276	2.195762	2.207781	2.220345	2.233465	2.247142	2.261371
.54	2.225331	2.235431	2.246024	2.257157	2.268871	2.281194	2.294146	2.307741	2.321984	2.336876	2.352405
.56	2.310030	2.320588	2.331739	2.343539	2.356026	2.369231	2.383174	2.397866	2.413311	2.429508	2.446448
.58	2.397222	2.408232	2.419951	2.432439	2.445736	2.459873	2.474869	2.490733	2.507467	2.525064	2.543501
.60	2.486902	2.498357	2.510654	2.523855	2.538002	2.553126	2.569244	2.586362	2.604478	2.623578	2.643667

TABLE 2. *The correction factor for the first overtone of flexural vibration of cylinders, T_2, as a function of the diameter to length ratio, d/l, and Poisson's ratio, μ.*

The values of T_2 for $d/l \leq 0.05$ were obtained by a graphical interpolation, as explained in the text. All other values were obtained from the computer solution.

d/l \ μ	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
0.00	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
.01	1.001352	1.001376	1.001399	1.001422	1.001445	1.001467	1.001489	1.001511	1.001533	1.001554	1.001576
.02	1.005404	1.005499	1.005592	1.005683	1.005774	1.005863	1.005952	1.006039	1.006126	1.006212	1.006297
.03	1.012144	1.012355	1.012564	1.012770	1.012973	1.013174	1.013373	1.013569	1.013763	1.013957	1.014149
.04	1.021552	1.021926	1.022295	1.022660	1.023019	1.023376	1.023728	1.024077	1.024421	1.024763	1.025103
.05	1.033605	1.034188	1.034760	1.035326	1.035885	1.036440	1.036987	1.037530	1.038064	1.038597	1.039128
.06	1.048267	1.049105	1.049924	1.050734	1.051537	1.052327	1.053108	1.053883	1.054653	1.055417	1.056178
.07	1.065511	1.066641	1.067750	1.068844	1.069926	1.070995	1.072054	1.073103	1.074146	1.075181	1.076211
.08	1.085296	1.086759	1.088198	1.089613	1.091015	1.092401	1.093775	1.095137	1.096491	1.097837	1.099177
.09	1.107583	1.109415	1.111219	1.112996	1.114756	1.116497	1.118223	1.119937	1.121641	1.123337	1.125025
.10	1.132330	1.134568	1.136772	1.138949	1.141103	1.143235	1.145352	1.147455	1.149547	1.151630	1.153705
.11	1.159496	1.162177	1.164820	1.167429	1.170012	1.172573	1.175115	1.177643	1.180161	1.182668	1.185170
.12	1.189048	1.192205	1.195318	1.198394	1.201441	1.204465	1.207470	1.210461	1.213439	1.216410	1.219376
.13	1.220947	1.224613	1.228230	1.231807	1.235353	1.238874	1.242376	1.245865	1.249343	1.252816	1.256285
.14	1.255166	1.259371	1.263523	1.267633	1.271711	1.275764	1.279800	1.283822	1.287838	1.291851	1.295863
.15	1.291674	1.296449	1.301168	1.305844	1.310487	1.315106	1.319710	1.324305	1.328896	1.333488	1.338084
.16	1.330448	1.335824	1.341140	1.346414	1.351656	1.356876	1.362084	1.367288	1.372493	1.377705	1.382928
.17	1.371471	1.377477	1.383422	1.389325	1.395199	1.401055	1.406905	1.412756	1.418616	1.424490	1.430382
.18	1.414728	1.421393	1.427999	1.434563	1.441104	1.447633	1.454162	1.460700	1.467255	1.473834	1.480442
.19	1.460212	1.467565	1.474861	1.482122	1.489364	1.496602	1.503850	1.511116	1.518411	1.525741	1.533111
.20	1.507918	1.515991	1.524010	1.532000	1.539980	1.547966	1.555973	1.564010	1.572090	1.580217	1.588400
.21	1.557849	1.566672	1.575448	1.584202	1.592958	1.601732	1.610540	1.619395	1.628306	1.637282	1.646330
.22	1.610014	1.619619	1.629185	1.638741	1.648312	1.657917	1.667572	1.677290	1.687084	1.696961	1.706930
.23	1.664427	1.674847	1.685239	1.695636	1.706063	1.716543	1.727092	1.737726	1.748455	1.759290	1.770238
.24	1.721106	1.732377	1.743634	1.754912	1.766240	1.777642	1.789136	1.800738	1.812460	1.824312	1.836303
.25	1.780080	1.792238	1.804400	1.816603	1.828880	1.841254	1.853748	1.866375	1.879151	1.892085	1.905185
.26	1.841381	1.854465	1.867575	1.880751	1.894027	1.907428	1.920979	1.934694	1.948588	1.962672	1.976953
.27	1.905048	1.919102	1.933207	1.947405	1.961734	1.976222	1.990892	2.005761	2.020844	2.036152	2.051692
.28	1.971129	1.986199	2.001348	2.016625	2.032066	2.047703	2.063560	2.079655	2.096003	2.112614	2.129495
.29	2.039679	2.055815	2.072063	2.088477	2.105095	2.121950	2.139068	2.156467	2.174161	2.192161	2.210474
.30	2.110759	2.128016	2.145425	2.163041	2.180906	2.199055	2.217513	2.236300	2.255430	2.274913	2.294753
.31	2.184442	2.202881	2.221516	2.240405	2.259595	2.279119	2.299006	2.319274	2.339937	2.361003	2.382477
.32	2.260809	2.280496	2.300430	2.320672	2.341270	2.362262	2.383673	2.405523	2.427824	2.450585	2.473806
.33	2.339949	2.360959	2.382273	2.403955	2.426056	2.448613	2.471654	2.495197	2.519253	2.543828	2.568923
.34	2.421965	2.444380	2.467163	2.490383	2.514090	2.538324	2.563111	2.588468	2.614407	2.640929	2.668033
.35	2.506968	2.530881	2.555234	2.580098	2.605527	2.631559	2.658220	2.685528	2.713488	2.742103	2.771366
.36	2.595085	2.620598	2.646633	2.673262	2.700540	2.728506	2.757184	2.786589	2.816726	2.847593	2.879180

TABLE 3. *The correction factor for the second overtone of flexural vibration, T_3, as a function of the diameter to length ratio, d/l, and Poisson's ratio, μ.*

The values of T_3 for $d/l \leq 0.04$ were obtained by a graphical interpolation as explained in the text. All other values were obtained from the computer solution.

d/l \ μ	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
0.00	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
.01	1.002609	1.002660	1.002710	1.002759	1.002808	1.002856	1.002903	1.002950	1.002997	1.003043	1.003089
.02	1.010421	1.010623	1.010823	1.011020	1.011214	1.011405	1.011594	1.011780	1.011968	1.012152	1.012335
.03	1.023390	1.023845	1.024290	1.024729	1.025164	1.025593	1.026015	1.026436	1.026855	1.027267	1.027676
.04	1.041446	1.042248	1.043034	1.043810	1.044578	1.045334	1.046082	1.046824	1.047563	1.048293	1.049019
.05	1.064495	1.065736	1.066954	1.068155	1.069343	1.070516	1.071678	1.072830	1.073974	1.075110	1.076240
.06	1.092436	1.094201	1.095938	1.097648	1.099340	1.101014	1.102673	1.104318	1.105953	1.107578	1.109196
.07	1.125153	1.127523	1.129857	1.132160	1.134438	1.136693	1.138930	1.141151	1.143360	1.145557	1.147746
.08	1.162526	1.165581	1.168592	1.171565	1.174505	1.177421	1.180314	1.183191	1.186053	1.188904	1.191747
.09	1.204448	1.208262	1.212021	1.215737	1.219416	1.223067	1.226696	1.230306	1.233902	1.237488	1.241067
.10	1.250814	1.255455	1.260034	1.264564	1.269056	1.273517	1.277955	1.282376	1.286786	1.291188	1.295587
.11	1.301530	1.307065	1.312532	1.317947	1.323321	1.328666	1.333990	1.339300	1.344603	1.349904	1.355208
.12	1.356516	1.363010	1.369431	1.375799	1.382127	1.388429	1.394714	1.400992	1.407270	1.413554	1.419850
.13	1.415707	1.423222	1.430664	1.438052	1.445405	1.452737	1.460061	1.467387	1.474723	1.482077	1.489456
.14	1.479051	1.487650	1.496175	1.504653	1.513102	1.521541	1.529982	1.538439	1.546921	1.555435	1.563990
.15	1.546513	1.556257	1.565933	1.575568	1.585187	1.594809	1.604449	1.614123	1.623839	1.633609	1.643439
.16	1.618072	1.629024	1.639915	1.650780	1.661643	1.672528	1.683453	1.694432	1.705478	1.716601	1.727810
.17	1.693723	1.705945	1.718120	1.730286	1.742473	1.754704	1.767001	1.779381	1.791856	1.804437	1.817134
.18	1.773475	1.787033	1.800562	1.814105	1.827696	1.841361	1.855124	1.869002	1.883011	1.897161	1.911461
.19	1.857353	1.872312	1.887269	1.902268	1.917349	1.932540	1.947868	1.963350	1.979003	1.994838	2.010865
.20	1.945396	1.961827	1.978286	1.994826	2.011487	2.028302	2.045298	2.062497	2.079912	2.097557	2.115440
.21	2.037655	2.055631	2.073675	2.091843	2.110181	2.128725	2.147502	2.166535	2.185839	2.205426	2.225305
.22	2.134200	2.153799	2.173513	2.193404	2.213523	2.233906	2.254584	2.275579	2.296906	2.318578	2.340601
.23	2.235112	2.256416	2.277894	2.299610	2.321620	2.343962	2.366670	2.389764	2.413260	2.437169	2.461496
.24	2.340488	2.363589	2.386928	2.410580	2.434600	2.459032	2.483908	2.509250	2.535071	2.561381	2.588182
.25	2.450441	2.475435	2.500745	2.526451	2.552613	2.579275	2.606470	2.634219	2.662535	2.691423	2.720883
.26	2.565101	2.592092	2.619491	2.647382	2.675827	2.704873	2.734552	2.764882	2.795876	2.827534	2.859853

TABLE 4. *A comparison of theoretical and experimental values of the correction factors* T_1, T_2, *and* T_3 *for the first 3 modes of flexural vibration of cylinders*

		Theoretical	Experimental	Percent difference
Specimen A, d/l=0.27871	T_1	1.3787	1.3790	−0.02
	T_2	2.0503	2.0498	+.02
	T_3	2.9814	2.9770	−.15
Specimen B, d/l=0.45039	T_1	1.9257	1.9251	+.03

The specimens were cut from a single bar of soft steel, and had a value of Poisson's ratio of 0.2879.

obtain and is probably in error by a larger amount than the other values. Also, it was necessary to extrapolate beyond the range of the tables to obtain this theoretical value. In spite of this, the indicated error is only 0.15 percent.

We may conclude from this evidence that no significant disagreement with experiment can be detected by the use of presently available techniques. It is therefore believed that the estimate of error given herein is realistic, not only with regard to the errors in the numerical computations, but also with regard to the equations used.

The author expresses his appreciation to Irene Stegun and Ruth Zucker for writing the IBM 704 computer program used in these computations.

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(Including papers in outside journals)

Infrared spectrum of hydrobromic acid, E. K. Plyler, *J. Research NBS* **64A,** *No. 5, 377 (1960).*
A precise measurement of the infrared fundamental bands of HBr^{79} and HBr^{81} has been made. The two band centers have been found to be 2558.94 and 2558.56 cm^{-1}, respectively. Rotational and vibrational constants have been calculated from the observed data. The constants are in good agreement with previous reported values. The centers of the two harmonic bands were used to calculate $\omega_e x_e$ and $\omega_e y_e$ and they were found to be 45.58 and 0.072 cm^{-1}, respectively for HBr^{79} and 45.56 and 0.072 cm^{-1} for HBr^{81}.

The spectrum of singly ionized atomic iodine (I II), W. C. Martin and C. H. Corliss, *J. Research NBS* **64A,** *No. 6, 443 (1960).*
The I II spectrum has been excited in electrodeless lamps and photographed from 655 to 11084 A. Wavelengths and estimated intensities are given for almost 2400 lines. A revision and extension of the earlier analyses of this spectrum has increased the number of known even levels from 43 to 124, and the number of odd levels from 55 to 190. New gJ-factors are given for 46 levels, and the previous designations of 40 levels are changed. Improved measurements in the vacuum ultraviolet region given a correction of 7.4 cm^{-1} to be subtracted from the values of all higher levels listed in *Atomic Energy Levels,* Vol. 3, (1958). The approximately 1800 classified lines now include all of the strongest lines. The $^1S_0$ of the ground configuration $5s^25p^4$ has been found, and this configuration has been fitted to intermediate coupling theory. Magnetic dipole transitions between levels of the ground configuration, $^3P_2$–$^1D_2$ (7282 A) and $^3P_1$–$^1S_0$ (4460 A), have been observed and their nature confirmed by the Zeeman effect. The line $5p^4\ ^3P_2$–$^1D_2$ shows hyperfine structure which is in approximate agreement with a theoretical calculation of the expected structure. New levels have been found for almost all higher configurations. All previously known series have been extended and new ones found. From one of the new series, $5p^3(^4S^\circ)5$–$12g\ ^5G^\circ_6$ the principal ionization energy for I II (154304 ±1, cm^{-1}) has been derived. The results of the analysis are compared with theoretical expectations in a number of cases.

The third spectrum of gold (Au III), L. Iglesias, *J. Research NBS* **64A,** *No. 6, 481 (1960).*
The spark spectrum of gold has been photographed in a helium atmosphere from 500 to 6600 A. About 500 lines have been assigned to the third spectrum, Au III, and separated from those belonging to other stages of ionization, by observation of the polarity of the lines. Sixty two levels have been found: 17 even levels, arising from the $5d^9$ and $5d^8$ 6*s* configurations; and 45 odd levels, belonging to the $5d^8$ 6*p* and $5d^7$ 6*s* 6*p* configurations. All of the expected levels from the configurations $5d^9$, $5d^8$ 6*s* and $5d^8$ 6*p* have been identified except for the very high terms based on the $5d^8$ (^1S) core of Au IV. With these levels it was possible to classify 256 lines.

Note on particle velocity in collisions between liquid drops and solids, O. G. Engel, *J. Research NBS* **64A,** *No. 6, 497 (1960).*
Equations are developed for plane-wave particle velocity produced in solid-against-liquid collisions. An explicit expression for the dimensionless coefficient α that appears in these equations is deduced.

Error analysis of a standard microwave phase shifter, G. E. Schafer and R. W. Beatty, *J. Research NBS* **64C,** *No. 4, 261 (1960).*
A standard microwave phase shifter has been proposed which utilizes an adjustable short circuit attached to a tunable three-arm waveguide junction. Two classes of errors considered are those due to imperfect tuning of the waveguide junction and those due to inaccuracies in determining the motion of the adjustable short circuit and the width of the waveguide, termed tuning and dimensional errors, respectively. Graphs are presented for estimating the scattering coefficient magnitudes needed to estimate limits of tuning error from observations of amplitude changes during the tuning procedure. Graphs are presented for estimating limits of dimensional error for WR–90 waveguide in the recommended frequency range of 8.2 to 12.4 kMc/s from tolerances of the axial motion of the short circuit and broad dimensions of the waveguide.

Electron resistivity studies on the Athabasca Glacier, Alberta, Canada, G. V. Keller and F. C. Frischknecht, *J. Research NBS* **64D,** *No. 5, 439 (1960).*
The use of electrical methods for measuring ice thickness and properties on the Athabasca Glacier, Alberta, Canada, has been studied by the U.S. Geological Survey. Two methods for measuring resistivity were tried: one, a conventional resistivity method in which current was introduced galvanically into the glacier through electrodes, and the other an electromagnetic method in which a wire loop laid on the ice was used to induce current flow. Results of the galvanic measurements showed large variations in the resistivity of the ice; in a surface layer several tens of feet thick the resistivity is between 0.3 and 1.0 megohm-meters, and under this layer, the resistivity of the ice is more than 10 megohm-meters. The resistivity of the surface ice is determined by its water content rather than by molecular resonance loss. The ice had no effect on the mutual coupling measurements in the frequency range from 100 to 10,000 cycles per second. As a consequence the electromagnetic data could be interpreted simply in terms of ice thickness and bedrock resistivity.

Amplitude distribution for radio signals reflected by meteor trails, A. D. Wheelon, *J. Research NBS* **64D,** *No. 5, 449 (1960).*
The probability distribution for the envelope of the received signal composed of reflections from many meteor trails is derived theoretically. Both the effects of numerous, small meteors and the residual reflections from infrequent, large meteors are treated simultaneously. For the particular example of exponential decay of initial spikes which are themselves distributed as the inverse square of their amplitudes, we find that the probability that the composite residual signal amplitude exceeds a prescribed level r is given by

$$P(R>r)=\frac{1}{\left[1+\frac{r^2}{(\nu\eta Q)^2}\right]^{1/2}}.$$

This function behaves as a Rayleigh distribution for small amplitude margins r. For the larger, less likely amplitudes it agrees with the result predicted by elementary analysis of isolated meteor reflections. Possible refinements of these results are also discussed.

Computation and measurement of the fading rate of moon-reflected UHF signals, S. J. Fricker, R. P. Ingalls, W. C. Mason, M. L. Stone, and D. W. Swift, *J. Research NBS* **64D,** *No. 5, 455 (1960).*
A method is described for predicting the fast fading rate of moon-reflected signals. It is based entirely upon considerations of the observer-moon positions and relative motions. Experimental results which are in good agreement with the computed fading rates have been obtained from a moon-reflection experiment at a frequency of 412 Mc/s. Some possible implications of this method of interpreting fading rates are given.

On the theory of wave propagation through a concentrically stratified troposphere with a smooth profile, H. Bremmer, *J. Research NBS* **64D,** *No. 5, 467 (1960).*
The W.K.B. approximation for the solution of the height-gain differential equation for a curved stratified troposphere is discussed in detail. The approximation depends mainly on a variable $u_l(r)$ which can be interpreted as the height dependent contribution of the phase for a field solution obtained by separation of variables. An expansion of $u_l(r)$ with the aid of partial integrations leads to further approximations which facilitate the determination of the eigen values, and of the amplitudes of the modes connected with the propagation problem. The influence of the refractive-index profile, if assumed as smooth, then appears to be restricted to a dependence on the surface values of this index and of its gradient insofar as propagation over the ground is concerned. Further, all height effects of elevated antennas can be expressed in terms of the distance to the corresponding radio horizon. This results in simple relations between the fields connected with two different refractive-index profiles, provided both profile coincide near the earth's surface.

Polarization and depression-angle dependence of radar terrain return, I. Katz and L. M. Spetner, *J. Research NBS* **64D,** *No. 5, 483 (1960).*
A study of recent experimental results on radar back scattering from land and sea surfaces indicate: (a) the polarization dependence of the normalized radar cross section, σ_0, of ocean surfaces cannot be explained by the usual "interference phenomenon" and (b) there is a distinct difference in the form of the depression-angle dependence in that σ_0 for "smooth" surfaces follows a negative exponential whereas σ_0 for "rough" surfaces drop off as the sine of the depression angle.

Methods of predicting the atmospheric bending of radio rays, B. R. Bean, G. D. Thayer, and B. A. Cahoon, *J. Research NBS* **64D,** *No. 5, 487 (1960).*
Three methods for predicting the bending of radio rays when the refractive index profile above the surface layer is unknown have been developed recently by the authors. These methods are: a statistical technique for refraction at high initial elevation angles, estimation of bending from an exponential model of atmospheric refractive index, and a modification of the exponential model to account for the heavily weighted effects of anomalous initial refractive index gradients at small initial elevation angles. Each model is dependent upon the value of the refractive index at ground level or, in the case of superrefraction, the additional knowledge of the refractive index gradient next to the earth's surface. Each method works best in a particular range of initial elevation angles or meteorological conditions. The height and angular ranges of application of each method are checked by comparison with values obtained from 77 diverse refractive index profiles representative of wide climatic variation. It is found that the use of the best of the three methods will always result in a prediction of the total atmospheric bending within 10 percent for initial elevation angles from zero to 10 mr and to within 4 percent for initial elevation angles greater than 17 mr (~1 degree).

Use of logarithmic frequency spacing in ionogram analysis, G. A. M. King, *J. Research NBS* **64D,** *No. 5, 501 (1960).*
The use of logarithmic frequency spacing in the reduction of ionograms to electron density profiles brings several advantages. Among them is the fact that, when computing factors for the analysis, one need not determine the group refractive index. Formulae involving only the phase refractive index are presented; for the ordinary component one exact and one approximate formula are given, while for the extraordinary component there is an approximate formula . . . valid over a wide range of geomagnetic latitudes. There is a brief discussion of quasi-longitudinal approximations to the extraordinary phase refractive index.

Guiding of whistlers in a homogeneous medium, R. L. Smith, *J. Research NBS* **64D,** *No. 5, 505 (1960).*
The velocity of energy flow of whistlers in a homogeneous medium is computed as a function of wave-normal angles. The maximum allowable cone of ray angles approaches 19°29′ at very low frequencies, decreases with frequency to a minimum of 11° at a wave frequency of one-fifth the gyrofrequency, then increases to 90° at the gyrofrequency. The velocity of energy flow departs markedly from the longitudinal value except at very low frequencies or very small wave-normal angles.

Shielding of transient electromagnetic signals by a thin conducting sheet, N. R. Zitron, *J. Research NBS* **64D,** *No. 5, 563 (1960).*
The shielding effect of a thin, horizontal imperfectly conducting sheet against the transient field of a vertical magnetic dipole when excited by a ramp function is investigated. The results are calculated by taking Laplace transforms of the frequency spectrum functions for the steady-state problem. The response to the ramp function is calculated and the significance of the results in shielding against surges is discussed.

Cylindrical antenna theory, R. H. Duncan and F. A. Hinchey, *J. Research NBS* **64D,** *No. 5, 569 (1960).*
A partial survey of cylindrical antenna theory pertaining to a tubular model with a narrow gap is presented. The survey includes discussion of the theories of Hallén, King and Middleton, Storm, and Zuhrt. A conceptual relation between theory and experiment is described. The latter part of the article is concerned with a new Fourier series solution of the Hallén equation. This solution is developed in such a way that the expansion coefficients are the unknowns of a system of linear equations. The elements of the coefficient matrix are given by a highly convergent series. Numerical results are given for half and full wavelength antennas with half length to radius ratios of 60 and 500. These results compare quite closely with those obtained from King-Middleton theory.

Vibration-rotation structure in absorption bands for the calibration of spectrometers from 2 to 16 microns, E. K. Plyler, A. Danti, L. R. Blaine, and E. D. Tidwell, *NBS Mono 16 (1960) 20 cents.*
Suitable bands of common gases have been tabulated and remeasured wherever necessary from 2 to 16 microns to obtain an accuracy of about 0.03 cm^{-1} throughout the region and to provide good calibrating points at frequent intervals. Some 600 rotation-vibration lines are illustrated in 20 spectrograms and wavenumbers are listed in companion tables with considerable intercomparison with worthy data obtained in other laboratories. The absorption bands were remeasured or calibrated by using either a precisely graduated grating circle or standard atomic lines with the fringe system formed by a Fabry-Perot interferometer. Characteristic features of the individual bands are discussed briefly and references to other publications are given. The substances used for calibration include H_2O, CO_2, CO, HCl, HBr, NH_3, C_2H_2, CH_4, N_2O, and polystyrene film. (Reprinted from the Journal of Research of the National Bureau of Standards—A. Physics and Chemistry, Vol. 64A, No. 1, January-February 1960.)

Dynamic measurements of the magnetoelastic properties of ferrites, V. E. Bottom, *NBS Tech. Note 49 (PB151408) (1960) $1.00.*
The relations between the mechanical and magnetic properties of a ferrite are derived for the small signal or reversible condition using thermodynamic principles. The equation of motion of a ferrite ring driven in its fundamental mode is set up and solved leading to the equivalent electrical circuit of the magnetostriction resonator. From mechanical measurements of the density and dimensions of the ring and electrical and frequency measurements of the resonator, the elastic modulus, permeability, magnetostriction coefficients and loss factors in the ferrite can be determined. Apparatus is proposed for performing the above measurements.

Magnetic drum directory and programming system for codesorting letter mail, P. C. Tosini, *NBS Tech. Note 50 (PB151409) (1960) $1.75.*
This report is an analysis and extension of the Rabinow Engineering Company's proposal for a magnetic drum file-directory and special purpose computer to be used in a test coded-address ("codesorting") letter-mail sorting system

scheduled for late 1960 installation at the Washington, D.C. Post Office. The report is composed of 6 major sections that successively examine the major considerations involved. A short introduction is presented first. Then the general problem of coding the fields composing addresses is discussed. This area is analyzed in terms of facilitating both the human coding process and the computer coded address-to-bin-number translation. Addresses are classified into two types: 'standard" and "non-standard". When coded, their fields must be unique and yet be sufficiently flexible to specify a variety of addresses. They must furthermore adequately specify some more common forms of misaddressed mail that is presently correctly sorted manually. A slightly amended form of Coding Plan E–1 is then presented and analyzed in terms of the conditions specified for coded addresses in general.

After the presentation of the coding area, the local mail sort translation program of the Rabinow proposal is discussed in detail. It is then extended to include the outgoing mail translation program and non-standard and special address forms. The information and instruction characters necessitated by the programs are listed in detail and the effects of the file-directory source document preparation determined. A coding keyboard is then suggested.

The programming is then reexamined and it is determined that all addresses can be treated as special cases of the standard address forms at an increased cost in memory space and with a saving in the special instruction symbols required. Finally, alternative methods of file-directory information storage are explored and evaluated in terms of memory space requirements, table look-up and directory access times, and conceptual simplicity.

Carrier frequency dependence of the basic transmission loss in tropospheric forward scatter propagation, K. A. Norton, *NBS Tech. Note 53* (*PB161554*) (*1960*) *$1.00.*

A further interpretation is given of certain Lincoln Laboratory data obtained in an experiment using scaled antennas as presented in a recent letter to the Proceedings of the I.R.E. from Bolgiano. This paper has four objectives: first, to clarify the significance of these data from the standpoint of the engineer developing long-range tropospheric scatter systems; second, to apply a further statistical analysis to these data; third, to consider their significance as regards the theory of radio propagation through a turbulent atmosphere; and fourth, to describe a suitable method for the measurement of the meteorological parameters entering the theory. Based on this analysis of the Lincoln Laboratory data, it is concluded that the carrier frequency dependence of the basic transmission loss cannot be variable from hour to hour.

Determination of a general index of effort in sorting mail by conventional methods, S. Henig, *NBS Tech. Note 54* (*PB161555*) (*1960*) *50 cents.*

The conventional method of sorting letters in stages is described. It is shown, through reference to available statistics, that about 90% of letters originating within the areas served by several large post offices will be involved in no more than five stages to be completely sorted to a carrier, firm or section of boxes. A method is presented for the accounting of each stage's individual letter readings which are defined as the units of measurement for the index of sorting effort. It is found that the upper bound average number of readings per letter for three large post offices varies from 3.16 to 2.98 and that these bounds are applicable to 96.2 to 92.4% of the local originating first class letters. It is anticipated that the values of such indexes for all large post offices will be consistently close to the range already determined.

Emission spectra of N_2, O_2, and NO molecules trapped in solid matrices, H. P. Broida and M. Peyron, *J. Chem. Phys.* **32,** *No. 4, 1068* (*1960*).

Molecular systems are observed in emission in solid products from a gas discharge trapped at liquid helium temperature. Previous tentative molecular assignments have been checked with the help of isotopic substitutions of oxygen and nitrogen. The Herzberg system (A bands) of oxygen ($A^3\Sigma_u^+ - X\ ^3\Sigma_g^-$) is analyzed and the molecular constants are derived for a molecule trapped in a nitrogen matrix. Another system (M bands) is attributed to the NO molecule ($^4\Pi - X^2\Pi$).

The components of power appearing in the harmonic analysis of a stationary process, M. M. Siddiqui, *presented at Symp. on Statistical Methods in Radio Wave Propagation, University of California, Los Angeles, June 18–20, 1958. Reprinted from "Statistical Methods in Radio Wave Propagation,"* (*Pergamon Press, New York, N.Y., 1960*).

Suppose that from a continuous record of a stationary process over a time interval T seconds, $2n + 1$ equally spaced readings are taken and the usual harmonic analysis is performed. Assuming the true frequencies present in the process to be $\ldots$, $1/(3T)$, $1/(2T)$, $1/T$, $2/T$, $\ldots$, cycles per second, it is shown that the power ascribed to the frequency j/T c/s consists of three components: (1) the true power in the frequency j/T c/s; (2) the powers in the frequencies $(2kn \pm j)/T$ c/s, $k=1,2, \ldots$; (3) a part of the powers in the frequencies $1/(mT)$ c/s, $m=2, 3, \ldots$.

Introduction—Can you measure it? L. V. Judson, Am. Soc. Tool & Mfgr. Engrs. Tech. Paper 239, Book 1, 60, 1 (1960). A paper which is the introductory paper in a Metrology Symposium. It indicates the need for precise and exact measurements and it treats two subsidiary questions: (1) Why measure? and (2) What has been the history of measurement?

Correction for systematic wavelength shift in atomic beam devices, R. L. Barger and K. G. Kessler, *J. Opt. Soc. Am.* **50,** *No. 4, 352* (*1960*).

Light emitted or absorbed by atoms in an atomic beam is shifted in frequency relative to the frequency for a stationary atom if the light ray is not normal to the trajectory of the atom. The magnitude of this shift is calculated and several devices which correct for this effect are described. For photoelectric recording of interference fringes, complete compensation can be achieved. In the case of photographic recording, it is shown that the residual error is negligible.

The error in prediction of *F2* maximum usable frequencies by world maps based on sunspot number, E. L. Crow and D. H. Zacharisen, *Section II, Radio Propagation Phenomenology, book, Statistical methods in radio wave propagation, p. 248* (*Pergamon Press, New York, N. Y., 1960*).

The feasibility of preparing semi-permanent world maps for predicting *F2* maximum usable frequencies based on sunspot number is studied. The components of variance of the prediction are estimated; they arise from the following four major sources of error: determining regressions of *F2* characteristics on sunspot number with limited years of data, the scatter of points about these regressions, drawing world contour maps with limited numbers of stations, predicting the future sunspot number. All four sources are found to contribute about equally to a total standard deviation of monthly median maximum usable frequency predicted five months in advance of about 1 and 3 Mc for 0 and 4000 km respectively. Hence the proposed maps are concluded to be feasible.

Bremsstrahlung linear polarization, J. W. Motz and R. C. Placious, *Il Nuovo Cimento* **15,** *Series X, 571* (*1960*).

This work presents a general quantitative description of the bremsstrahlung linear polarization on the basis of original experimental data and available theoretical calculations. The results give the dependence of the polarization on *a*) the initial electron kinetic energy, T_0 in a range from 10^{-2} to 10^3 MeV, *b*) the photon energy in a range from 0.1 T_0 to T_0, *c*) the photon emission angle in a range from 0 to 180 degrees, and *d*) the atomic number of the target in a range from 4 to 79. The experimental data were obtained for a range of electron energies from 0.05 to 1.0 MeV with beryllium, aluminum, and gold targets. Theoretical estimates of the polarization were calculated from the Sommerfeld-Kirkpatrick-Wiedmann results for the non-relativistic energy region, the Olsen-Maximon results for the extreme relativistic energy region, and the Gluckstern-Hull (Born approximation) results for the intermediate energy region. Final results are expressed in terms of the peak polarization and the corresponding peak angle as a function of the electron and photon energies, and best estimates of the polarization are given on the basis of the combined experimental and theroretical data.

Geomagnetic disturbances and velocity of slow-drift solar radio bursts, M. G. Wood and C. S. Warwick, Nature **184,** No. 4697, 1471 (1959).

Frequency drift rates were measured on records of the Fort Davis dynamic spectrum analyzer for 48 Type II (slow-drift) solar radio noise bursts. The change with frequency of the frequency drift rate is found to differ for two groups of bursts: those followed by geomagnetic disturbance and those followed by no disturbance. This change in drift rate is interpreted as acceleration of the radio source as it moves outward through the solar corona. Bursts followed by geomagnetic disturbance determine a rate that corresponds to greater positive acceleration of the source.

The outlook for machine translation, F. L. Alt, *Proc. Western Joint Computer Conf., Volume 17, p. 203, San Francisco, Calif., (May 1960).*

The current status of the major research projects concerned with the use of computing machines for the translation of natural languages (e.g., Russian to English) is surveyed. None of these projects has as yet reached the stage of fully automatic translation of satisfactory quality. The difficulties encountered fall into two main classes: semantic and syntactic. Several methods of approach have been used to overcome them; these are surveyed, and the approach of the National Bureau of Standards is described in more detail. It appears likely that within a few years automatic translation of fair quality may be possible, at a cost substantially below that of human translation. The major cost elements—initial development of machine programs, cost of machines and of operation—are analyzed in their dependence on the method of translation and on the type of equipment used.

Statistical evaluation of interlaboratory cement tests, J. R. Crandall and R. L. Blaine, *Am. Soc. Testing Materials Proc.* **59,** *1129 (1959).*

A cooperative series of physical and chemical tests was made by 103 laboratories on twelve samples of cement during the period of one year. Results of tests by all laboratories on each property were plotted in scatter-diagrams according to the Youden method to enable the participating laboratories to quickly evaluate their results. The results were treated statistically. Both statistical and graphic methods were employed to indicate the precision of the test methods and the extent of laboratory bias. The lack of precision with some of the tests indicated a need for improved test methods.

The work of each of the various laboratories was evaluated by a rating system. A large number of laboratories obtained results that were in good agreement but a few laboratories showed poor agreement with the larger group. Many of the discrepant laboratories did not improve during the one-year period the test program was active.

Discrepant results obtained by a few laboratories inflated appreciably the standard deviations among laboratories for the various tests.

The need for a continuing reference sample program was indicated.

Effect of notch geometry on tensile properties of annealed titanium at *100°, 25°, −78°* and *−196°* C, G. W. Geil and N. L. Carwile, *Am. Soc. Testing Materials Proc.* **59,** *985 (1959).*

The combined effects of low temperatures and geometry of a circumferential notch on the tensile behavior of initially annealed commercially-pure titanium was investigated. The notch geometry was varied by changes in its depth and root radius; the notch depth, in terms of the cross-sectional area removed in machining the notch, ranged from 5 to 87 percent and the root radius ranged from 0.005 to 2.0 in. True stress-true strain data were obtained to the initiation of the fracture of specimens at 100°, 25°, −78° and −196° C and representative true stress-true strain curves are presented. The relationships of selected strength and ductility indices to the notch geometry and to the accompanying triaxialities and stress concentration factors are presented graphically and discussed in some detail in the text.

The wavefront reversing interferometer, J. B. Saunders, *Colloquium on Optics and Metrology, Sponsored by the Intern. Commission of Optics, Brussels, Belgium, May 8–9, 1958. Publ. in Optics in Metrology, p. 227 (Pergamon Press, New York, N. Y., 1960).*

The wavefront reversing interferometer is designed to test the shape of wavefronts. If a wavefront is characteristic of the shape of an optical element, or combination of elements, the evaluation of the wavefront is indirectly an evaluation of the element. It serves to measure the shapes of reflecting surfaces, the aberrations of lenses, and the variations of optical paths in a beam, such as inhomogeneity of the medium through which the light traveled. A unique method of analyzing the data is given. An instrument suitable for testing lenses and mirrors, by projecting the interference fringes onto a screen so that they may be observed by an audience, is described.

Spectra emitted from rare gas-oxygen solids during electron bombardment, L. J. Schoen and H. P. Broida, *J. Chem. Phys.* **32,** *No. 4, 1184 (1960).*

Light emission from rare gas solids containing small amounts of oxygen has been excited by an electron bombardment technique. The Herzberg bands of oxygen and a line group believed to be the $^1S \rightarrow {}^1D$ transition of the neutral oxygen atom form the strongest features of the observed spectra. In a neon matrix, the Second Negative system of O_2^+ has been identified. The effect of the various solids on the vibrational structure of the Herzberg bands is discussed.

Accurate microwave wavemeters with convenient calibration tables, H. E Bussey and A. J. Estin, *Rev. Sci. Instr.* **31,** *410 (1960).*

Accurate and convenient microwave cavity wavemeters are described that are suitable for many precise physical measurements. High Q's were attained by refined construction techniques. At 9000 Mc a precision of 0.02 Mc was obtained. The absolute accuracy also may be very high after strains in the metal have stabilized. A calibration table containing 10^4 entries, easily formed by means of a high speed computer, makes the wavemeters convenient to use. The curve fitting method, accurate to one in 10^6, is described.

The Stokes flow problem for a class of axially symmetric bodies, L. E. Payne and W. H. Pell, *J. Fluid Mech.* **7,** *Pt. 4, 529 (1960).*

The Stokes flow problem is concerned with fluid motion about an obstacle when the motion is such that inertial effects can be neglected. This problem is considered here for the case in which the obstacle (or configuration of obstacles) has an axis of symmetry, and the flow at distant points is uniform and parallel to this axis. The differential equation for the stream function ψ then assumes the form $L^2{}_{-1}\psi = 0$, where L_{-1} is the operator which occurs in axially symmetric flows of the incompressible ideal fluid. This is a particular case of the fundamental operator of A. Weinstein's generalized axially symmetric potential theory. Using the results of this theory and theorems regarding representations of the solutions of repeated operator equations, the authors (1) give a general expression for the drag of an axially symmetric configuration in Stokes flow, and (2) indicate a procedure for the determination of the stream function. The stream function is found for the particular case of the lens-shaped body.

Explicit calculation of the drag is difficult for the general lens, without recourse to numerical procedures, but is relatively easy in the case of the hemispherical cup. As examples illustrative of their procedures, the authors briefly consider three Stokes flow problems whose solutions have been given previously.

Analysis of two-factor classifications with respect to life tests, M. Zelen, *Book, Contributions to Probability and Statistics, Paper 42, p. 508 (Stanford Univ. Press, Stanford, Calif., 1960).*

Recently, much attention has been given to a study of the exponential distribution with particular emphasis on applications in life testing, e.g., Epstein and Sobel, Gupta, and Sobel. These papers have mainly considered techniques and procedures associated with observations from a single ex-

ponential distribution. This paper outlines procedures to be used for analyzing the results of a two-way classification with respect to the scale factor of the exponential distribution. The test procedures are based on likelihood ratio tests and have their analogs in the usual analysis-of-variance tests for main effects and interactions. Approximations are given for tests based on small samples. These same procedures carry over directly to a two-way analysis of variance, if the quantities to be analyzed are variance estimates following chi-square distributions.

Microwave spectrum of trans-crotononitrile, V. W. Laurie, *J. Chem. Phys.* **32**, *No. 5, 1588 (1960).*
The microwave spectrum of ethyl cyanide has been studied in the region from 17–36 kMc. Both parallel and perpendicular transitions have been assigned. Rotational constants (Mc) for the ground vibrational state are $a_0=27\ 663.30$, $b_0=4714.14$, $c_0=4235.14$, $D_J=0.0035$, $D_{JK}=-0.0496$. From Stark effects the dipole moment is calculated to be $\mu a=3.78\ D$, $\mu b=1.38\ D$, $\mu=4.02\ D$. Hyperfine splittings lead to a value of −3.3 Mc for the quadrupole coupling of the N^{14} nucleus along the *a* axis. Rotational transitions of molecules in several excited vibrational states have also been measured. A Coriolis interaction between the torsional mode and the in-plane CCN bend has been found and is discussed. Splittings of transitions of molecules in the first excited torsional state show that the barrier hindering internal rotation of the methyl group is 3.05 kcal/mole.

Disk distribution of flares associated with certain radio bursts, C. Warwick, *Proc. Symp. Physical Processes in the Sun-Earth Environment, July 20–21, 1959 (Ottawa, Defence Research Telecommunications Establishment), DRTE Pub. No. 1025, p. 249 (1960).*
Flares associated in time with U-bursts and with slowdrift bursts have an unusual distribution over the solar disk that may be caused by a directive effect in the radio emission. Such an effect, making the observation of these radio bursts more probable from a source near the limb of the sun than from a source near the center, could arise in coupling processes either between plasma waves and electromagnetic radiation, or between the two modes of electromagnetic radiation.

List of Titles

Journal of Research, Section 64A, No. 5, September–October 1960. 70 cents.
Infrared spectrum of hydrobromic acid. E. K. Plyler. (See above abstracts.)
Determination of the value of the faraday with a silver-perchloric acid coulometer. D. N. Craig, J. I. Hoffman, C. A. Law, and W. J. Hamer.
Systems silver iodide-sodium iodide and silver iodide-potassium iodide. G. Burley and H. E. Kissinger.
Conformations of the pyranoid sugars. III. Infrared absorption spectra of some acetylated aldopyranosides. R. S. Tipson and H. S. Isbell.
Dissociation constant of 4-aminopyridinium ion in water from 0° to 50° C and related thermodynamic quantities. R. G. Bates and H. B. Hetzer.
Tritium-labeled compounds. VI. Alditols-*1-t* and alditols-*2-t*. H. L. Frush, H. S. Isbell, and A. J. Fatiadi.

Journal of Research, 64A, No. 6, November-December 1960. 70 cents.
The spectrum of singly ionized atomic iodine (I II). W. C. Martin and C. H. Corliss. (See above abstracts.)
The third spectrum of gold (Au III). L. Iglesias. (See above abstracts.)
Tolerances for layer thicknesses in dielectric multilayer coatings and interference filters. K. D. Mielenz.
Note on particle velocity in collisions between liquid drops and solids. O. G. Engel. (See above abstracts.)
Resistance of white sapphire and hot-pressed alumina to collision with liquid drops. O. G. Engel.
Note on the thermal degradation of polytetrafluoroethylene as a first-order reaction. S. L. Madorsky and S. Straus.
Heat of formation of titanium trichloride. W. H. Johnson, A. A. Gilliland, and E. J. Prosen.
Heat of formation of decaborane. W. H. Johnson, M. V. Kilday, and E. J. Prosen.
Ultra low-conductivity water by electrophoretic ion exclusion. W. Haller and H. C. Duecker.
Spectrophotometric determination of the ionization constant of dimethylpicric acid (2,4,6-trinitro-3,5-xylenol) in water at 25° C. M. M. Davis, M. Paabo, and R. A. Robinson.
Spectrophotometric determination of the ionization constant of 2,4,6-trinitro-m-cresol in water at 25° C. M. M. Davis and M. Paabo.
Method for the separation of titanium, zirconium, iron, and aluminum from one another and for their subsequent determination. T. J. Murphy, W. S. Clabaugh, and R. Gilchrist.

Journal of Research, Section 64C, No. 4, October–December 1960. 75 cents.
Error analysis of a standard microwave phase shifter. G. E. Schafer and R. W. Beatty. (See above abstracts.)
A method of controlling the effect of resistance in the link circuit of the Thomson or Kelvin double bridge. D. Ramaley.
Automatic precise recording of temperature. G. S. Ross and H. D. Dixon.
Gimbal device to minimize the effects of off-center loading on balance pans. H. A. Bowman and L. B. Macurdy.
Response of microchemical balances to changes in relative humidity. H. E. Almer.
Chemical changes occurring during the weathering of two coating-grade asphalts. S. H. Greenfeld.
Characteristics of fifteen coating-grade asphalts. S. H. Greenfeld.

Journal of Research, Section 64D, No. 5, September-October 1960. 70 cents.
ELF electric fields from thunderstorms. A. D. Watt.
Field strength measurements in fresh water. G. S. Saran and G. Held.
Electrical resistivity studies on the Athabasca Glacier, Alberta, Canada. G. V. Keller and F. C. Frischknecht. (See above abstracts.)
Amplitude distribution for radio signals reflected by meteor trails. A. D. Wheelon. (See above abstracts.)
Computation and measurement of the fading rate of moon-reflected UHF signals. S. J. Fricker, R. P. Ingalls, W. C. Mason, M. L. Stone, and D. W. Swift. (See above abstracts.)
On the theory of wave propagation through a concentrically stratified troposphere with a smooth profile. H. Bremmer. (See above abstracts.)
Polarization and depression-angle dependence of radar terrain return. I. Katz and L. M. Spetner. (See above abstracts.)
Methods of predicting the atmospheric bending of radio rays. B. R. Bean, G. D. Thayer, and B. A. Cahoon. (See above abstracts.)
Loss in channel capacity resulting from starting delay in meteor-burst communication. G. R. Sugar.
Elementary considerations of the effects of multipath propagation in meteor-burst communication. G. R. Sugar, R. J. Carpenter, and G. R. Ochs.
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